MATH501 Analysis
Fall 2021-2022
Cheat Sheet

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These are the summaries I made from when I was taking the MATH501 Analysis course from Süleyman Kağan Samurkaş during the Fall 2021-2022 term. We followed *Real analysis for graduate students* by Richard F. Bass, and the section numbers are in accordance with the sections there.

Contents

1. Preliminaries

1.1. Notation & Terminology

Complement: The complement of a subset $A \subset X$, denoted A^{c}, is given as

$$A^{\mathsf{c}} = X - A \text{ and so } A - B = A \cap B^{\mathsf{c}}$$

Symmetric Difference: The symmetric difference of two subsets $A, B \subset X$ is given as

$$A \triangle B = (A - B) \cup (B - A)$$

Increasing Sets: For an indexed family of sets $\{A_i\}$, we write $A_i \uparrow$ to denote that

$$A_1 \subset A_2 \subset A_3 \subset \ldots$$

We further write $A_i \uparrow A$ to mean

$$A_1 \subset A_2 \subset A_3 \subset \ldots \qquad \& \qquad \bigcup_{i=1}^{\infty} A_i = A$$

Decreasing Sets: For an indexed family of sets $\{A_i\}$, we write $A_i \downarrow$ to denote that

$$A_1 \supset A_2 \supset A_3 \supset \ldots$$

We further write $A_i \downarrow A$ to mean

$$A_1 \supset A_2 \supset A_3 \supset \ldots \qquad \& \qquad \bigcap_{i=1}^{\infty} A_i = A$$

Positive & Negative Parts: For a real number x, the positive and negative parts of x, denoted x^+ and x^- respectively, are defined as

$$x^+ = \max\{x, 0\} \qquad \& \qquad x^- = \max\{-x, 0\}$$

This definition is used to define the positive and negative parts of a real function.

Limsup & Liminf: The limsup and liminf *of a sequence* $\{a_i\}$ are defined as follows:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_n \sup_{m\geq n} a_m$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_n \inf_{m\geq n} a_m$$

The limsup and liminf *of a function* $f(x)$ are defined as follows:

$$\limsup_{x\to a} f(x) = \inf_{\delta<0} \sup_{|x-a|<\delta} f(x)$$

$$\limsup_{x\to a} f(x) = \sup_{\delta<0} \inf_{|x-a|<\delta} f(x)$$

Left and Right Limits: For a function with domain $\mathbb{R}$, the left and right limits at x are denoted as follows:

$$f(x+) = \lim_{y\to x^+} f(y) \qquad \text{and} \qquad f(x-) = \lim_{y\to x^-} f(y)$$

1.2. Some Undergraduate Maths

Metric Spaces: A set X is a metric space if there exists a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

Open Ball: In a metric space X, the open ball around $x \in X$ with radius r is defined as follows:

$$B(x; r) = \{y \in X : d(x, y < r)\}$$

Interior & Closure: For a subset $A \subset X$ of a metric space X, the interior A^o of A and the closure $\overline{A}$ of A are defined as follows:

$$A^o = \{x \in X : \exists r_x > 0 \ B(x; r_x) \subset A\}$$

$$\overline{A} = \{x \in X : \forall r > 0 \ B(x; r) \cap A \neq \varnothing\}$$

Opennes & Closedness: The set A is called open if $A = A^o$, closed if $A = \overline{A}$.

Opennes in $\mathbb{R}$: An open subset $G \subseteq \mathbb{R}$ can always be written as a disjoint and countable union of open intervals (proven via Zorn's Lemma).

Open Cover & Compactness: An open cover of a subset $K \subset X$ is a non-empty collection of open sets $\{G_\alpha\}_{\alpha \in I}$ such that

$$K \subset \bigcup_{\alpha \in I} G_\alpha$$

A subset $K \subset X$ is called compact if every open cover of K has an finite subcover, i.e.

$$\forall \{G_\alpha\} \text{ with } K \subset \bigcup G_\alpha, \ \exists G_1, G_2, ..., G_n \text{ with } K \subset \bigcup_{i=1}^{n} G_i$$

Heine-Borel Theorem: A subset $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Extreme Value Theorem: If K is compact and $f : K \to \mathbb{R}$ is continuous, then there exist $x_m, x_M \in K$ such that

$$f(x_m) = \inf_{x \in K} f(x) \qquad \text{and} \qquad f(x_M) = \sup_{x \in K} f(x)$$

Cauchy Sequences & Completeness: A sequence $\{a_n\}_{n \in \mathbb{N}}$ in a metric space X is called Cauchy if

$$\forall \varepsilon > 0 \exists N > 0 \text{ such that } \forall n, m > N \ d(a_n, a_m) < \varepsilon$$

The verbal interpretation of this statement is that any tail of the sequence gets as close to one another as we want. However, Cauchy sequences need not converge: We say that a metric space X is complete if every Cauchy sequence in X has a limit in X.

Hausdorff Property: Any two distinct points in a metric space can be separated by non-intersecting open sets. This is called the Hausdorff property.

Normed Spaces: X is a normed linear space if it is a vector space with a function $\|\cdot\| : X \to [0, \infty)$ satisfying the following properties:

1. $\|x\| > 0$ for all $x \neq 0_X$, and $\|x\| = 0$ if and only if $x = 0_X$.
2. $\|cx\| = |c|\|x\|$ for all $x \in X$ and $c \in F$, where for us $F = \mathbb{R}$ or $F = \mathbb{C}$.

3. $\|x+y\| \le \|x\| + \|y\|$, which is also called the triangle inequality.

A norm on a normed space X induces a metric as $d(x,y) = \|x-y\|$. A metric on a *vector space* induces a norm $\|x\| = d(x,0)$.

Partial Orders: A relation $\le \subset X \times X$ on X is a partial ordering if

1. For all $x \in X$, $x \le x$.
2. If $x \le y$ and $y \le x$, then $x = y$.
3. If $x \le y$ and $y \le z$, then $x \le z$.

A partially-ordered set is called a poset in short.

Total Order: A partial order is called a total order if any two elements can be compared, so for all $x, y \in X$ we have either $x \le y$ or $y \le x$.
A totally-ordered subset of a poset is called a chain.

Zorn's Lemma: For any non-empty poset X, if any non-empty chain has an upper bound in X, then there is at least one maximal element in X.

Continuity of Increasing Functions: Suppose $f : \mathbb{R} \to \mathbb{R}$ is an increasing function. Then both $f(x+)$ and $f(x-)$ exist, and the set of x's where f is not continuous must be countable. (Remember: The discontinuities can only be jump discontinuities!)

2. Families of Sets

2.1. Algebras & σ-algebras

Algebra & σ-algebra: An algebra is a collection $\mathcal{A}$ of subsets of X such that

1. $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$,
2. If $A \in \mathcal{A}$, then $A^{\mathsf{c}} \in \mathcal{A}$,
3. If $A_1, A_2, ..., A_n \in \mathcal{A}$, then $\bigcup_{i=1}^n A_i,\ \bigcap_{i=1}^n A_i \in \mathcal{A}$.

$\mathcal{A}$ is further a σ-algebra if we have not only finite but also countable unions and intersections:

4. If $A_1, A_2, ..., \in \mathcal{A}$, then $\bigcup_{i=1}^\infty A_i, \bigcap_{i=1}^\infty A_i \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a measurable space, and $A \in \mathcal{A}$ is called measurable if $A \in \mathcal{A}$.

Intersection of σ-algebras: If $\{\mathcal{A}_\alpha\}$ is a collection of σ-algebras over X, then $\bigcap_\alpha \mathcal{A}_\alpha$ is again a σ-algebra.

Generated σ-algebra: For $\mathcal{C} \subset \mathcal{P}(X)$, the σ-algebra generated by $\mathcal{C}$, denoted $\sigma(\mathcal{C})$, is defined as

$$\sigma(\mathcal{C}) = \bigcap \{\mathcal{A}_\alpha : \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra containing } \mathcal{C}\}$$

With this definition, we have the following two properties:

- If $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subset \sigma(\mathcal{C}_2)$.
- $\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C})$

Borel σ-algebra: For a metric space X and $\mathcal{G}$ the collection of open sets of X induced by its metric, the σ-algebra $\mathcal{B} = \sigma(\mathcal{G})$ is called the Borel σ-algebra on X. The sets $A \in \mathcal{B}$ are called Borel-measurable.

Generators of $\mathcal{B}$ on $\mathbb{R}$: For $X = \mathbb{R}$, $\mathcal{B}$ is generated by the following collections:

- $\mathcal{C}_1 = \{(a, b) : a, b \in \mathbb{R}\}$
- $\mathcal{C}_2 = \{[a, b] : a, b \in \mathbb{R}\}$
- $\mathcal{C}_3 = \{(a, b] : a, b \in \mathbb{R}\}$
- $\mathcal{C}_4 = \{(a, \infty) : a \in \mathbb{R}\}$

2.2. The Monotone Class Theorem

Monotone Class: A monotone class is a collection $\mathcal{M}$ of subsets of X such that

1. If $A_1, A_2, A_3, ... \in \mathcal{M}$ and $A_i \uparrow A$, then $A \in \mathcal{M}$,
2. If $A_1, A_2, A_3, ... \in \mathcal{M}$ and $A_i \downarrow A$, then $A \in \mathcal{M}$.

The intersection and generative properties stated above for σ-algebras hold for monotone classes as well.

Monotone Class Theorem: For an algebra $\mathcal{A}_0$, let $\mathcal{A} = \sigma(\mathcal{A}_0)$ and $\mathcal{M}$ be the monotone class generated by $\mathcal{A}_0$. Then we have

$$\mathcal{A} = \mathcal{M}$$

3. Measures

Measure: Let $X, \mathcal{A}$ be a measurable space. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$, and
2. If $A_i \in \mathcal{A}$ for $i = 1, 2, 3, ...$ are pairwise-disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$. This property is called countable additivity. If the identity holds for a finite set instead of a countable one, then it is called finite additivity.

The triple $(X, \mathcal{A}, \mu)$ is then called a measure space.

Properties: For a fixed measure space $(X, \mathcal{A}, \mu)$, the followings hold:

1. If $A, B \in \mathcal{A}$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
2. If $A_i \in \mathcal{A}$ for $i = 1, 2, 3, ...$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
3. Suppose $A_i \in \mathcal{A}$ and $A_i \uparrow A$. Then $\mu(A) = \lim_{i\to\infty} \mu(A_i)$.
4. Suppose $A_i \in \mathcal{A}$ and $A_i \downarrow A$ AND $\mu(A_1) < \infty$. Then $\mu(A) = \lim_{i\to\infty} \mu(A_i)$.

Some Useful Constructions:

Countable union to disjoint countable union: Suppose we have $A_i \in \mathcal{A}$ for all i and $A = \bigcup_{i=1}^{\infty} A_i$. Then construct the sets B_i in the following way:

$$B_1 = A_1 \qquad \text{and} \qquad B_n = A_n - (A_{n-1} \cup ... \cup A_1) = A_n - \bigcup_{i=1}^{n-1} A_i$$

This way, B_i are disjoint, measurable sets, satisfying

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \qquad \text{and} \qquad \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Countable union to increasing sequence: Suppose we have $A_i \in \mathcal{A}$ for all i and $A = \bigcup_{i=1}^{\infty} A_i$. Then construct the sets B_i in the following way:

$$B_1 = A_1 \qquad \text{and} \qquad B_n = \bigcup_{i=1}^{n} A_i$$

This way, B_i are measurable sets satisfying

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \qquad \text{and} \qquad B_i \uparrow A$$

Countable intersection to decreasing sequence: Suppose we have $A_i \in \mathcal{A}$ for all i and $A = \bigcap_{i=1}^{\infty} A_i$. Then construct the sets B_i in the following way:

$$B_1 = A_1 \qquad \text{and} \qquad B_n = \bigcap_{i=1}^{n} A_i$$

This way, B_i are measurable sets satisfying

$$\bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \qquad \text{and} \qquad B_i \downarrow A$$

Finiteness & σ-Finiteness: A measure μ is called a finite measure if $\mu(X) < \infty$. A measure μ is called σ-finite if there exists some countable collection of sets $E_i \in \mathcal{A}$ with $\mu(E_i) < \infty$ and $\bigcup_{i=1}^{\infty} E_i = X$.

Null Sets & Completeness: For a measure space $(x, \mathcal{A}, \mu)$, a subset $N \subset X$, is called a null set if there exists some $A \in \mathcal{A}$ with $N \subseteq A$ and $\mu(A) = 0$. Notice that a null set need not be measurable.

If a measure space $(X, \mathcal{A}, \mu)$ contains all null sets, it is said to be complete.

Completion: The completion of a σ-algebra $\mathcal{A}$ is the smallest complete σ-algebra $\overline{\mathcal{A}}$ that contains $\mathcal{A}$, such that $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space, which is obtained by setting

$$\overline{\mathcal{A}} = \sigma(\mathcal{A} \cup \mathcal{N}) \qquad \text{and} \qquad \overline{\mu}\big|_{\mathcal{A}} = \mu$$

where $\mathcal{N}$ is the collection of all null sets.

4. Construction of Measures

4.1. Outer Measures

Outer Measure: For a set X, an outer measure on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\mu^*(\varnothing) = 0$,
2. If $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$, and
3. For all countable selection of subsets $A_1, A_2, A_3, ... \in \mathcal{P}(X)$, $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

A set N is a null set with respect to μ^* if $\mu^*(N) = 0$.

Length Function & Induced Outer Measure: Suppose $\mathcal{C}$ is a collection of subsets of X, such that $\varnothing \in \mathcal{C}$ and there exist $E_1, E_2, E_3, ... \in \mathcal{C}$ with $\bigcup E_i = X$. Suppose further that there exists a function $l : \mathcal{C} \to [0, \infty]$ with $l(\varnothing) = 0$. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ as

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(C_i) : C_i \in \mathcal{C} \text{ and } A \subset \bigcup_{i=1}^{\infty} C_i \right\}$$

This μ^* is then an outer measure.

Outer Measurability: For an outer measure μ^*, a set $A \subseteq X$ is said to be μ^*-measurable if for all $E \subseteq X$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^{\mathsf{c}})$$

Induced Complete Measure: For an outer measure μ^*, the collection

$$\mathcal{A} = \{ A \subseteq X : \ A \text{ is } \mu^*\text{-measurable } \}$$

is a σ-algebra and $\mu = \mu^*\big|_{\mathcal{A}}$ is a complete measure.

4.2. Lebesgue-Stieltjes Measures

Lebesgue-Stieltjes Measures: Let $X = \mathbb{R}$, $\mathcal{C} = \{(a, b] : a, b \in \mathbb{R}\}$ and let $\alpha : \mathbb{R} \to \mathbb{R}$ be a right-continuous and increasing function. Define a length function as

$$l((a, b]) = \begin{cases} \alpha(b) - \alpha(a) & , a < b \\ 0 & , a \geq b \end{cases}$$

This length function and the collection $\mathcal{C}$ satisfy the conditions above, and so the following is a valid outer metric:

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(C_i) : C_i \in \mathcal{C} \text{ and } A \subset \bigcup_{i=1}^{\infty} C_i \right\}$$

and so The collection of m^*-measurable functions, say $\mathcal{M}$, is a σ-algebra, and $m = m^*\big|_{\mathcal{M}}$ is a complete measure. This measure m is called the Lebesgue-Stieltjes measure (associated to the given α function).

Lebesgue Measure: In the special case where $\alpha(x) = x$, the induced σ-algebra is called the Lebesgue σ-algebra, and the measure is called the Lebesgue measure.

Useful Lemma: Let $J_k = (a_k, b_k)$ for $k = 1, 2, ..., n$ be a finite collection of open intervals covering a closed interval $[c, d]$, and α a right-continuous increasing function. Then we have

$$\alpha(d) - \alpha(c) \leq \sum_{i=1}^{n} \alpha(b_k) - \alpha(a_k)$$

Agreement on the Generators: Let $e, f \in \mathbb{R}$ with $e \leq f$. Then the outer measure m^* agrees with the length function l, so

$$m^*((e, f]) = l((e, f]) = \alpha(f) - \alpha(e)$$

Relation to the Borel σ-algebra: Every set in the Borel σ-algebra is m^*-measurable, in other words, the Borel σ-algebra is contained in any σ-algebra associated to a Lebesgue-Stieltjes (or Lebesgue) measure.

$$\mathcal{B} \subseteq \mathcal{L}$$

This also implies that there might exist (and indeed there does exist) some set that is Lebesgue-Stieltjes (or Lebesgue) measurable but not Borel measurable.

4.3. Examples and Related Results

Cantor Set: Define the following sequence of sets:

$$\begin{aligned} C_0 &= [0,1] \\ C_1 &= C_0 - (1/3, 2/3) = [0, 1/3] \cup [2/3, 1] \\ C_2 &= C_1 - ((1/9, 2/9) \cup (7/9, 8/9)) = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1] \\ &\dots \end{aligned}$$

and set $C = \bigcap_{i=0}^{\infty} C_i$. C is a closed and bounded set constructed by removing the "middle third" of each interval iteratively, and it can alternatively be expressed as

$$C = \{x \in [0,1] : x = \underbrace{(0, a_1 a_2 a_3 a_4 \cdots)_3}_{\text{base-3 expression}} \text{ such that } a_i \in \{0, 2\}\}$$

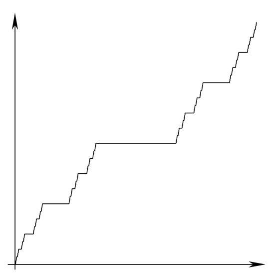
Now some facts about C:

- From the alternative description, we see that C is in bijection with the complete $[0,1]$ as well: Just set 2's to 1's and read the number in base 2. So, the set is uncountable.
- Every point in C is a limit point, so for all $x \in C$, there exists some sequence $\{a_i\} \subseteq C - \{x\}$ such that $a_i \to x$. We can also see this from the alternative definition: For all n, take the sequence element a_n as the first n numerals after the ternary point if the sequence contains infinitely many 2's, namely $a_n = (0, a_1 a_2 \dots a_n 00 \dots)_3$. If it contains finitely many 2's with the final 2 at, say a_k, set the k-th numeral to 0, add a 2 after the k-th numeral with some zeroes in between: $a_n = (0, a_1 a_2 \dots a_{k-1} 0 \dots 020 \dots)_3$. It's easy to see that this sequence satisfies the limit point condition.
- C contains no interval. If it did, it's interior would contain a middle-third of some interval in the construction and that would have been removed. This holds for any interval, and the inside of each interval is iteratively removed. On the other hand, we can also see it via the base-3 representation: If it did contain an interval, say $[x, y]$ where $x, y \in C$ and $x \neq y$, it would also contain a number containing a 1 numeral in its base-3 representation necessarily, which is a contradiction to the definition of C.

Now, what is the Lebesgue measure of C? We see that $C_n \downarrow C$ and $m(C_0) = 1 < \infty$, so we can say $m(C) = \lim(C_n)$. As $m(C_n) = (2/3)^n$, this means that $m(C) = \lim(2/3)^n = 0$. So, we have found an uncountable set with measure 0!

Cantor-Lebesgue Function: Define a function $f_0 : [0,1] \to [0,1]$ using the removed parts of the Cantor set:

$$f_0(x) = \begin{cases} 1/2 & \text{if } x \in (1/3, 2/3) \\ 1/4 & \text{if } x \in (1/9, 2/9) \\ 3/4 & \text{if } x \in (7/9, 8/9) \\ \dots \end{cases}$$



and using f_0, define $f_{CL} : [0,1] \to [0,1]$ as

$$f_{CL} f(x) = \begin{cases} \inf\{f_0(y) : y \geq x, y \notin C\} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Now, we have $f_{CL}\big|_{[0,1]-C} = f_0$ and f is always increasing, so f_{CL} can only have jump discontinuities. That it cannot have either: Say it is discontinuous over an interval. This interval must contain a number of the form $k/2^n$, which is in the image of f_0, and hence that of f_{CL}, which yields a contradiction. So, f is an increasing continuous function, and as it is piecewise-constant on $[0,1] - C$, it can only increase on C.

Generalized Cantor Set: Instead of always removing the "middle-third," we can also choose to remove the "middle-a_i" section at each i-th step, where $a_i \in (0,1)$. For example, if we choose $a_i = 1/4$ for all i, the resultant set will have measure $1/2$, and again will be closed, uncountable, without intervals and every point in it will be a limit point.

Playing inside $0 \le x \le 1$: For a Lebesgue-measurable subset $A \subseteq [0,1]$, we have

1. For all $\varepsilon > 0$, there exists an open $G \supseteq A$ such that $m(G - A) < \varepsilon$.
2. For all $\varepsilon > 0$, there exists a closed $F \subseteq A$ such that $m(A - F) < \varepsilon$.
3. There exists some $H \supseteq A$ such that $G_i \downarrow H$ with each G_i open and $m(H - A) = 0$.
4. There exists some $J \subseteq A$ such that $F_i \downarrow H$ with each F_i closed and $m(A - K) = 0$.

G_δ and F_σ sets: A set A is called a G_δ set if it is a countable intersection of open sets. It is called a F_σ set if it is a countable union of closed sets.

4.4. Nonmeasurable Sets

Non-outer Measurable Sets: Let m^* be the outer measure associated to $l : \mathcal{C} = \{(a,b] : a,b \in \mathbb{R}\} \to [0,\infty]$, $l((a,b]) = b - a$ whenever $(a,b] \neq \varnothing$. Then the collection of m^*-measurable sets is not equal to $\mathcal{P}(\mathbb{R})$. This means that there must exist some subset of $\mathbb{R}$ that is not m^* measurable. Define an equivalence relation $\sim$ on $[0,1]$ as follows:

$$x \sim y \iff x - y \in \mathbb{Q}$$

Let A be a set of representatives from $[0,1]/\sim$. The claim is that A is not Lebesgue-measurable. Assume to the contrary that it is. Then because $A \subseteq [0,1]$, we can write the following:

$$[0,1] \subseteq \bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A + q) \subseteq [-1,2]$$

The first subset relation is true because we are shifting the set of representatives via all the possible rational shifts that can occur within $[0,1]$, and the second subset relation is easy to see. Then, because all the shifts of A under different q values are disjoint, we have

$$1 = m^*([0,1]) \le \sum_{q \in [-1,1] \cap \mathbb{Q}} m^*(A+q) \le m^*([-1,2]) = 3$$

Now, because a shift by any q preserves the length of the set, it must also preserve the outer measure m^* of the set, so

$$1 = m^*([0,1]) \le \sum_{q \in [-1,1] \cap \mathbb{Q}} \underbrace{m^*(A+q)}_{=m^*(A)} \le m^*([-1,2]) = 3$$

$$1 \le \sum_{q \in [-1,1] \cap \mathbb{Q}} m^*(A) \le 3$$

Now, what is $m^*(A)$? It must be either 0, or positive. If it is positive, the sum yields 0, which is strictly smaller than 1, a contradiction. If it is any positive real, then the sum is unbounded, which contradicts the second inequality. Thus, A has no outer measure, and thus is not Lebesgue-measurable.

4.5. Carathédory Extension Theorem

Measure on an Algebra: For an algebra (but not necessarily a σ-algebra) $\mathcal{A}_0$, a function $l : \mathcal{A}_0 \to [0, \infty]$ is a measure if

1. $l(\varnothing) = 0$, and
2. If $A_i \in \mathcal{A}$ for $i = 1, 2, 3, ...$ are pairwise-disjoint with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_0$, then $l\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} l(A_i)$. (Notice that the inclusion of the union is forced, not due to the properties of $\mathcal{A}_0$.)

Carathédory Extension Theorem: Suppose $\mathcal{A}_0$ is an algebra and $l : \mathcal{A}_0 \to [0, \infty]$ is a measure as above. Let

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(A_i) : A_i \in \mathcal{A}_0 \quad \text{and} \quad A \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

Then

1. μ^* is an outer measure.
2. $\mu^*(A) = l(A)$ whenever $A \in \mathcal{A}_0$.
3. Every set in $\mathcal{A}_0$ and every μ^*-null set is μ^*-measurable.
4. If l is σ-finite, meaning there exists some K_i sets each with finite measure such that $\bigcup_{i=0}^{\infty} K_i = \mathbb{R}$, then l uniquely extends to $\sigma(\mathcal{A}_0)$.

5. Measurable Functions

5.1. Measurability

Measurability of Functions: Let $(X, \mathcal{A})$ be a measure space. A function $f : X \to \mathbb{R}$ is measurable or $\mathcal{A}$-measurable if the inverse image of a Borel-measurable set is $\mathcal{A}$-measurable. Equivalently,

- $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$,
- $f^{-1}((a, b)) \in \mathcal{A}$ for all $a, b \in \mathbb{R}$,
- $f^{-1}((a, b]) \in \mathcal{A}$ for all $a, b \in \mathbb{R}$,
- $f^{-1}([a, b)) \in \mathcal{A}$ for all $a, b \in \mathbb{R}$,

or the most usefully,

- $f^{-1}((a, \infty)) \in \mathcal{A}$ for all $a, b \in \mathbb{R}$,

Typical Examples: Constant functions $f(x) = c$ are always measurable.
The characteristic function χ_A of a set A is measurable if and only if A is measurable, i.e. $A \in \mathcal{A}$.
If the σ-algebra $\mathcal{A}$ contains the open subsets of X, then any continuous function $f : X \to \mathbb{R}$ is measurable.

Operations on Measurable Functions: Given that the functions $f, g : X \to \mathbb{R}$ are both measurable, so are the following functions:

$$cf \text{ where } c \in \mathbb{R} \qquad f + g \qquad f - g \qquad fg \qquad \max\{f, g\} \qquad \min\{f, g\}$$

Sequences of Measurable Functions: If f_i's are real-valued measurable functions for all $i = 1, 2, ...$, then the following functions are also measurable:

$$\inf_i f_i \qquad \sup_i f_i \qquad \liminf_{i\to\infty} f_i \qquad \limsup_{i\to\infty} f_i$$

Lebesgue- and Borel-Measurability of Functions: Let X be a metric space, $\mathcal{B}_X$ the Borel σ-algebra on X (σ-algebra generated by the open sets of X, $\mathcal{B}$ denotes the σ-algebra on $\mathbb{R}$) and $\mathcal{L}$ be the Lebesgue σ-algebra on $\mathbb{R}$ (σ-algebra of all m^*-measurable sets where m^* is the induced outer metric of $\alpha(x) = x$). If

- $f : (X, \mathcal{B}_X) \to (\mathbb{R}, \mathcal{B})$ is measurable, then f is called Borel-measurable.
- $f : (X, \mathcal{L}) \to (\mathbb{R}, \mathcal{B})$ is measurable, then f is called Lebesgue-measurable.

Notice how the range σ-algebra is always Borel.

Monotonity: If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel-measurable.

A Lebesgue-measurable but not Borel-measurable set: Define the function F using f_{CL} the Cantor-Lebesgue function as follows:

$$F(y) = \inf\{x \in [0, 1] : y \leq f_{CL}(x)\}$$

This $F : [0, 1] \to C$ is strictly increasing, where C is the Cantor set.
Now take A as the set we have previously constructed via the equivalence class representatives, which is not Lebesgue-measurable, and consider its image $F(A) \subseteq C$. Because $m(C) = 0$ and the Lebesgue measure is complete, any subset of C is measurable and has measure 0. In particular, $F(A)$ is Lebesgue measurable with $m(F(A)) = 0$.
Now assume that $F(A)$ is Borel-measurable. Because F is strictly increasing, it is Borel-measurable (being monotone). Because we assume $F(A)$ to be Borel-measurable, we would conclude that $F^{-1}(F(A)) = A$ is Borel-measurable as well. However, we also know that $\mathcal{B} \subseteq \mathcal{L}$ and that $A \notin \mathcal{L}$, so A cannot be in $\mathcal{B}$, which yields the contradiction.
Therefore we conclude that

$$\mathcal{B} \subsetneq \mathcal{L}$$

5.2. Approximation of Functions

Simple Functions: A simple function on a σ-measurable space $(x, \mathcal{A})$ is of the form

$$s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) \qquad \text{where} \qquad a_i \in \mathbb{R}, E_i \in \mathcal{A}$$

Being a linear sum of characteristic functions, simple functions are always measurable.

Approximation via Simple Functions: Suppose f is a non-negative, measurable function. Then, there exists a sequence of non-negative and measurable simple functions s_n which increase to f, meaning for all $x \in X$, we have $s_n(x) \nearrow f(x)$.

Proof via Pointing a Finger: Define the following sets using f:

$$A_{in} = \left\{x : \frac{i-1}{2^n} \leq f(n < \frac{i}{2^n})\right\} \qquad \text{for} \qquad n = 1, 2, ... \text{ and } i = 1, 2, ..., n2^n$$

$$B_n = \{x : n \leq f(x)\} \qquad \text{for} \qquad n = 1, 2, ...$$

and define

$$s_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{A_{in}}(x) + n\chi_{B_n}$$

This approximation satisfies the requirement above.

6. The Lebesgue Integral

Lebesgue Integral ... Let $(X, \mathcal{A}, \mu)$ be a measure space.

... of a Simple Function: For a non-negative simple function $s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$, define its Lebesgue integral as follows:

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$$

This definition is indeed well-defined. Here, we use the convention that if $a_i = 0$ and $\mu(E_i) = \infty$ for some i, we say $a_i \mu(E_i) = 0$.

... of a Positive Measurable Function: If f is any measurable function, define its Lebesgue integral as

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu : 0 \leq s \leq f, \quad s \text{ is a measurable non-negative simple function} \right\}$$

... of a Measurable Function: For a general measurable function f, define the positive and negative parts of f as

$$f^+ = \max\{f, 0\} \qquad \text{and} \qquad f^- = \max\{-f, 0\}$$

These are both positive and measurable functions, so their Lebesgue integrals are as defined above. Unless both of their individual integrals are infinity, define the integral of f as

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

If both of these integrals are infinity, then the integral of f is not defined.

... over a set A: The integral of f over a set A is defined as

$$\int_A f \, d\mu = \int f \chi_A \, d\mu$$

Integrability: If f is measurable and we have

$$\int |f| \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu < \infty$$

then f is called integrable.

Properties of the Lebesgue Integral:

Squeezing Between Constants: If f is a measurable function with $0 \leq a \leq f(x) \leq b$ for all $x \in X$ for non-negative $a, b \in \mathbb{R}$ and the measure of the whole space is finite, i.e. $\mu(X) < \infty$, then

$$a\mu(X) = \int a \, d\mu \leq \int f \, d\mu \leq \int b \, d\mu = b\mu(X)$$

Compared Functions: If f and g are measurable and integrable functions with $0 \leq f(x) \leq g(x)$ for all $x \in X$, then

$$\int f \, d\mu \leq g \, d\mu$$

Homogenity over Positive Constants: If f is measurable and integrable, and $c \geq 0$ is a constant, then

$$\int cf \, d\mu = c \int f \, d\mu$$

Integration over Sets of Measure 0: For a subset $A \subseteq X$ with $\mu(A) = 0$ and a non-negative measurable function f, we have

$$\int_A f \, d\mu = \int f \chi_A \, d\mu = 0$$

7. Limit Theorems

7.1. Monotone Convergence Theorem

Monotone Convergence Theorem (MCT): Let $\{f_n\}$ be a sequence of

- non-negative,
- measurable,
- increasing functions, i.e. $f_1(x) \leq f_2(x) \leq f_2(x) \leq \dots$ for all x,
- with limit $\lim_{n\to\infty} f_n(x) = f(x)$ for a measurable function f.

Then, we have

$$\int f_n \; d\mu \longrightarrow \int f \; d\mu \qquad \text{i.e.} \qquad \lim_{n\to\infty} \int f_n \; d\mu = \int \lim_{n\to\infty} f_n \; d\mu$$

Note that this result holds a.e.

7.2. Linearity of the Integral

Basic Linearity: Let f and g be non-negative and measurable OR only integrable functions. Then

$$\int f + g = \int f + \int g$$

The two functions being only measurable is not enough.

Improved Properties of the Lebesgue Integral: The previously proved properties of the Lebesgue integral are now extended to be valid for not only non-negative numbers/functions:

Squeezing Between Constants: If f is a measurable function with $a \leq f(x) \leq b$ for all $x \in X$ for ANY $a, b \in \mathbb{R}$ and the measure of the whole space is finite, i.e. $\mu(X) < \infty$, then

$$a\mu(X) = \int a \; d\mu \leq \int f \; d\mu \leq \int b \; d\mu = b\mu(X)$$

Compared Functions: If f and g are measurable and integrable NOT NECESSARILY NON-NEGATIVE functions with $\leq f(x) \leq g(x)$ for all $x \in X$, then

$$\int f \; d\mu \leq g \; d\mu$$

Homogenity over Positive Constants: If f is measurable and integrable, and c is ANY constant, then

$$\int cf \; d\mu = c \int f \; d\mu$$

Integration over Sets of Measure 0: For a subset $A \subseteq X$ with $\mu(A) = 0$ and ANY measurable function f, we have

$$\int_A f \; d\mu = \int f\chi_A \; d\mu = 0$$

Integral of a Function Series: Suppose f_n are a sequence of measurable and non-negative functions. Then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

Triangle Inequality: For an integrable function f, we have

$$\left|\int f\right| \leq \int |f|$$

7.3. Fatou's Lemma

Fatou's Lemma: Suppose f_n's form a sequence of measurable functions. Then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

A typical use: Suppose $f_n \to f$ and we have $\sup_n \int |f_n| \leq K < \infty$. Then, $|f_n| \to |f|$ and $\int |f| \leq K$.

Proof.

$$\int \int |f| = \int \liminf_{n\to\infty} |f_n| \leq \liminf_{n\to\infty} \int |f_n| \qquad \text{(Fatou's Lemma)}$$

$$\leq \sup_n \int |f_n| \leq K$$

□

7.4. Dominated Convergence Theorem

Dominated Convergence Theorem (DCT): Suppose f_n are measurable with $f_n \to f$, and there exists a non-negative and integrable function g with $|f_n| \leq g$. Then

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f$$

The g here is said to dominate the function sequence f_n.
Note that this result holds a.e.

8. Properties of Lebesgue Integrals

8.1. Criteria for a Function to be Zero a.e.

Zero Integral of ...

... a Non-Negative Function: Suppose $f \geq 0$ is measurable with

$$\int f \, d\mu = 0$$

Then $f = 0$ a.e.

... an Integrable Function on $A \in \mathcal{A}$: Suppose f is an integrable function, with

$$\int_A f \, d\mu = 0$$

for any measurable set A. Then $f = 0$ a.e.

... in the Lebesgue Measure: Let m be the Lebesgue measure and $a \in \mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable, with

$$\int_a^x f(y)\ m(dy) = \int_{[a,x]} f\ dm = 0$$

for all $x \in \mathbb{R}$. Then $f = 0$ a.e.

8.2. An Approximation Result

Continuous Approximation: Let f be a Lebesgue-measurable and integrable function on $\mathbb{R}$, and $\varepsilon > 0$. Then, there exists a continuous function g with compact support $\operatorname{supp}(g) = \{x : g(x) \neq 0\}$ such that $\int |f - g| \leq \varepsilon$.
Any integrable function can be approximated with a continuous function with compact support.

10. Types of Convergence

Convergence ... Let f_n be a sequence of functions, in a measure space with measure μ.

... a.e.: f_n is said to converge to some f a.e. if if

$$\mu(\{x : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$$

... in Measure: f_n is said to converge in measure to f if for all $\varepsilon > 0$, if

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0$$

... in L^p: Let $1 \leq p < \infty$. Then f is said to converge to f in L^p if

$$\lim_{n\to\infty} \int |f_n - f|^p\ d\mu = 0$$

In measure and a.e. Convergence under Finite Measures: Suppose μ is a finite measure. Then

1. If $f_n \to f$ a.e., then $f_n \to f$ in measure.
2. If $f_n \to f$ in measure, then there is a subsequence f_{n_j} such that $f_{n_j} \to f$ a.e.

Chebyshev's Inequality: Let $1 \leq p < \infty$ and $a > 0$. Then

$$\mu(\{x : |f(x)| \geq a\}) \leq \frac{\int |f|^p\ d\mu}{a^p}$$

L^p and in Measure Convergence: If $f_n \to f$ in L^p, then $f_n \to f$ in measure.

11. Product Measures

11.1. Product σ-algebras

Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two measure spaces.

Product σ-algebra: Define the product σ-algebra as

$$\mathcal{A} \times \mathcal{B} = \sigma(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\})$$

$A \times B$ where $A \in \mathcal{A}, B \in \mathcal{B}$ is called a measurable rectangle.

Sections: Given some $E \subset X \times Y$, define the following functions:

$$s_x(E) = \{y \in Y : (x, y) \in E\} \forall x \in X \qquad (x\text{-section})$$
$$t_y(E) = \{x \in X : (x, y) \in E\} \forall y \in Y \qquad (y\text{-section})$$

Measurable Sections: Let $E \in \mathcal{A} \times \mathcal{B}$. Then,

- $s_x(E) \in \mathcal{B}$ for all $x \in X$, and
- $t_y(E) \in \mathcal{A}$ for all $y \in Y$.

Measurable Univariate Functions: Suppose f is a $\mathcal{A} \times \mathcal{B}$-measurable function, also called jointly measurable. Then,

- $k(y) = f(x^*, y)$ is $\mathcal{B}$-measurable for all $x^* \in X$, and
- $h(x) = f(x, y^*)$ is $\mathcal{A}$-measurable for all $y^* \in Y$.

Measure of Sections: Suppose μ and ν are σ-finite, and let $E \in \mathcal{A} \times \mathcal{B}$. Now define

$$h(x) = \nu(s_x(E)) \qquad \text{and} \qquad k(y) = \mu(t_y(E))$$

Then h is $\mathcal{A}$-measurable and k is $\mathcal{B}$-measurable, with

$$\int h(x)\mu(dx) = \int k(y)\nu(dy)$$

These two integrals correspond to the following double integral:

$$\iint \chi_E(x, y)\mu(dx)\nu(dy) = \iint \chi_E(x, y)\nu(dy)\mu(dx)$$

Product Measure: Through this fact, we can define the product measure $\mu \times \nu$:

$$\mu \times \nu(E) = \int h(x)\mu(dx) = \int k(y)\nu(dy) \quad \forall E \in \mathcal{A} \times \mathcal{B}$$

For measurable a measurable rectangle $A \times B$ gives the intuitive $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ result.

Completeness of the Product Measure: Even if μ and ν are individually complete, the product measure $\mu \times \nu$ may be incomplete: Take $\mu = \nu = m$ and $A \times 0$. $A \times 0$ is a $m \times m$-null set, but is not $\mathcal{L} \times \mathcal{L}$-measurable.

n-dimensional Lebesgue Measure: The n-dimensional Lebesgue measure is defined as the completion of the measure space $(\mathbb{R}^n, \mathcal{L} \times \cdots \times \mathcal{L}, m \times \cdots \times m)$.

11.2. The Fubini Theorem

Fubini-Tonelli Theorem: Suppose $f : X \times Y \to \mathbb{R}$ is a jointly measurable function, and μ and ν are σ-finite measures. If either

a. f is non-negative, or

b. f is integrable, i.e. $\iint |f| \; d(\mu \times \nu) < \infty$

then the followings hold:

1. Measurable Univariates: The functions

$$y \mapsto f(x^*, y) \forall x^* \in X \qquad \text{and} \qquad x \mapsto f(x, y^*) \forall y^* \in Y$$

are measurable within their own spaces.

2. Measurable Marginals: The functions

$$h(x) = \int f(x,y) \nu(dy) \qquad \text{and} \qquad x \mapsto k(y) = \int f(x,y) \mu(dx)$$

are measurable within their own spaces.

3. Order of Integration: The order of integration does not matter, i.e.

$$\begin{aligned} \iint f(x,y) \; d(\mu \times \nu)(x,y) &= \int \left[\int f(x,y) \; \mu(dx) \right] \nu(dy) \\ &= \int \left[\int f(x,y) \; \nu(dy) \right] \mu(dx) \end{aligned}$$

12. Signed Measures

12.1. Positive and Negative Sets

Signed Measure: Let $\mathcal{A}$ be a σ-algebra. A signed measure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to (-\infty, \infty)$ such that

- $\mu(\varnothing) = 0$, and
- if $A_1, A_2, \dots \in \mathcal{A}$ are disjoint, then

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i)$$

where the series converges absolutely if the sum is finite.

We require absolute convergence so the order of summation does not matter.
The measures we have seen so far are sometimes called as positive measures.

Positive, Negative and Null Sets: Let μ be a signed measure. A set A is said to be

positive if $A \in \mathcal{A}$ and $\mu(B) \geq 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$.
negative if $A \in \mathcal{A}$ and $\mu(B) \leq 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$.
a null set if $\mu(B) = 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$.

A null set for a positive measure is also a null set in this sense.

Countable Union: By a similar reasoning done in the (previous) positive measure case, we again have

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \lim_{n \to \infty} \mu \left(\bigcup_{i=1}^{n} A_i \right)$$

Negative Subsets: Let μ be a signed measure, and $E \in \mathcal{A}$. If $\mu(E) < 0$, then there exists a measurable and negative subset $F \subseteq E$.

12.2. Hahn Decomposition Theorem

Hahn Decomposition Theorem: Let μ be a signed measure. The followings then hold:

Positive/Negative Partition: There exists $P, N \subseteq X$ with $P \sqcup N = X$, where P is a positive set and N is a negative set.

Almost Uniqueness: If P' and N' are another such pair, then $P \triangle P' = N \triangle N'$ is a null set with respect to μ.

Measures of the Partition: If μ is not a positive measure, then $\mu(N) < 0$.
If $-\mu$ is not a positive measure ("negative" measure), then $\mu(P) > 0$.

Support of a Measure: Given a positive measure μ, we say that μ is supported on a set $Y \subseteq X$ if for all $A \in \mathcal{A}$ with $A \subseteq Y^c$ we have $\mu(A) = 0$.

Mutually Singular Measures: Two positive measures μ and ν are said to be mutually singular if there exist $A, B \subseteq X$ with $A \sqcup B = X$, μ is supported on A and ν is supported on B. In that case, we write $\mu \perp \nu$.

12.3. Jordan Decomposition Theorem

Jordan Decomposition Theorem: Let μ be a signed measure on $(X, \mathcal{A})$. Then, there exist *unique* positive measures μ^+ and μ^- such that

$$\mu = \mu^+ - \mu^- \qquad \text{and} \qquad \mu^+ \perp \mu^-$$

These two positive measures are defined as

$$\mu^+(A) = \mu(A \cap P) \qquad \text{and} \qquad \mu^-(A) = -\mu(A \cap N)$$

for all $A \in \mathcal{A}$, where P and N are the positive and negative sets, respectively, from the Hahn decomposition of μ.

13. The Radon-Nikodym Theorem

13.1. Absolute Continuity

Absolute Continuity: A measure ν is said to be absolutely continuous with respect to a measure μ if

$$\mu(A) = 0 \Rightarrow \nu(A) = 0$$

denoted $\nu \ll \mu$.

For Finite Measures: If ν is a finite measure, then $\nu \ll \mu$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall A \in \mathcal{A}\ \mu(A) < \delta \Rightarrow \nu(A) < \varepsilon$$

13.2. The Main Theorem

Finite Positive Measures: Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{A})$. Then either $\mu \perp \nu$ or there exists $\varepsilon > 0$ and $G \in \mathcal{A}$ such that $\mu(G) > 0$ and G is positive for the signed measure $\nu - \varepsilon\mu$.

Radon-Nikodym Theorem: Let μ and ν are positive σ-finite measures on $(X, \mathcal{A})$ with $\nu \ll \mu$. Then, there exists an $\mathcal{A}$-measurable function f such that

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$. If g is another such function, then $f = g$ μ-a.e.

13.3. Lebesgue Decomposition Theorem

Lebesgue Decomposition Theorem: Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{A})$. Then, there exist positive measures λ and ρ such that

$$\nu = \lambda + \rho \qquad \text{and} \qquad \rho \ll \mu$$

15. L^p Spaces

15.1. Norms

L^p Norm: For $1 \le p < \infty$, the L^p norm of a function f is defined as

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}$$

For $p = \infty$, the L^∞ norm is defined as

$$\|f\|_\infty = \inf\{M \ge 0 : \mu(\{x : |f(x)| \ge M\}) = 0\}$$

If no such M exists, then we say $\|f\|_\infty = \infty$. $\|f\|_\infty$ is the smallest number M such that $|f| < M$ a.e.

Conjugate Exponent: For $1 \le p \le \infty$, define the conjugate exponent q of p as follows:

- If $1 < p < \infty$, then let q be uniquely defined such that $1/p + 1/q = 1$.
- If $p = 1$, then $p = \infty$.
- If $p = \infty$, then $q = 1$.

Hölder's Inequality: Let $1 \le p \le \infty$ and q be the conjugate exponent of p. Then for measurable functions f and g, we have

$$\int |fg| \, d\mu \le \|f\|_q \|g\|_q$$

For $p = q = 2$, the inequality becomes the Cauchy-Schwartz Inequality.

Minkowski's Inequality: For $1 \le p \le \infty$ and measurable functions f and g, we have

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

This is also called, very reasonably, triangle inequality for the L^p norm.

L^p **Normed Linear Space:** Define an equivalence relation $\sim$ as follows

$$f \sim g \iff f - g = 0 \text{ a.e.}$$

Then define

$$L^p = \{f : \|f\|_p < \infty\}/\sim$$

Under this definition and the norm $\|\cdot\|_p$, $(L^p, \|\cdot\|_p)$ becomes a valid linear normed space:

1. $\|f\|_p \geq 0$ and $\|f\|_p = 0$ if and only if $f = 0$ a.e., i.e. $f \sim 0$
2. $\|f + g\|_p \leq \|f\|_p + \|g\|_p$
3. $\|\alpha f\| = |\alpha| \|f\|_p$

The important thing to keep in mind when talking about the L^p space is that when we take some $f \in L^p$, we are not taking one function but an *equivalence class* of functions.

Essential Infimum & Essential Supremum:

$$\text{ess sup } f = \inf\{M : \mu(\{x : f(x) > M\}) = 0\}$$
$$\text{ess inf } f = \sup\{m : \mu(\{x : f(x) < m\}) = 0\}$$

These definitions are very much like infimum/supremum, the only difference is that they are "blind" to differences over null sets.

15.2. Completeness

Completeness: For $1 \leq p \leq \infty$, the normed linear space $(L^p, \|\cdot\|_p)$ is complete.

Dense Subset: The set of continuous functions with compact support is dense in $L^p(\mathbb{R})$ for $1 \leq p \lneq \infty$.
This means that any $f \in \mathcal{L}^p$ can be approximated with infinite precision by a continuous function with compact support.

Exlusion of $p = \infty$: Notice that $p = \infty$ is excluded from this statement. Here's why: Take $1 \in L^p(\mathbb{R})$. For any continuous function f with compact support, we have $\|1 - f\|_\infty \geq 1$, so the approximation error cannot be made infinitely small under the given norm.

A special case: The set of continuous functions on $[a, b] \subset \mathbb{R}$ is dense in $L^2([a, b])$.

15.3. Convolutions

In this subsection, all functions are defined on $\mathbb{R}^n$ and the measure is the Lebesgue measure on $\mathbb{R}^n$.

Convolution: For measurable functions f and g, the convolution is defined as follows:

$$(f * g)(x) = \int f(x - y)g(y)dy$$

if the integral exists.

Measurability: The function $f(x-y)g(y)$ is jointly measurable, and so the convolution integral $\int f(x - y)g(y)dy$ results in a measurable function by the Fubini-Tonelli theorem.

Commutativity: The convolution operation is commutative, i.e. $f * g = g * f$.

Inequalities with L^p Norms:

- If $f, g \in L^1$, then $f * g \in L^1$ with

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

- If $f \in L^1$ and $g \in L^p$ where $1 < p \leq \infty$, then $f * g \in L^p$ with

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Mollification: The procedure of approximating any function in L^p with a smooth function is called mollification. The approximation of $f \in L^p$ is obtained via a convolution operation with a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with compact support that is infinitely differentiable, non-negative and with integral equal to 1.
Notice that if φ is such a function, so is $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ for $\varepsilon > 0$.

Properties of the Mollification: Suppose $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$.

1. **Differentiability:** For all $\varepsilon > 0$, $f * \varphi_\varepsilon$ is infinitely differentiable.
2. **Derivatives:** For any non-negative integers $\alpha_1, \ldots, \alpha_n$ the partial derivatives of $f * \varphi_\varepsilon$ are given as follows:

$$\frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}}(f * \varphi_\varepsilon) = f * \left(\frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}} \varphi_\varepsilon \right)$$

3. **Approximation:** $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0^+$.
4. **Uniform Convergence:** If f is continuous, then $f * \varphi_\varepsilon \to f$ uniformly on compact sets as $\varepsilon \to 0^+$.
5. **Convergence in L^p:** For $1 \leq p < \infty$, $f * \varphi_\varepsilon \to f$ in L^p as $\varepsilon \to 0^+$.