MATH497 Introduction to Hilbert Spaces
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Cheat Sheet

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Contents

0. Preliminaries

Separability means that there exists a countably infinite sequence that is dense in the space, meaning every non-empty open interval contains one element of the sequence.

0.1. Linear Spaces

Bases (Hamel) exist in any linear space (via Zorn's lemma): Any linearly independent subset can be extended to a basis in a linear space.

Complemented spaces: If $E = F \oplus E$, then F is a complement of G and G is a complement of F (For E, F, G linear spaces). The direct sum here means that the intersection $F \cap G$ is is trivial, i.e. $= \{\theta\}$.
Every linear subspace is complemented: Take a basis in a subspace F and extend it to a basis in E. Say the span of the "added" elements is G. Then $E = F \oplus G$.

Projections: $P : E \to F$ is a projection if $P^2 = P$ for a linear map P. In this case $F = P(E)$ is a linear space and $(I - P)(E) = G$ is a complement to F.
Conversely, every direct sum gives a projection: If $E = F \oplus G$, then define $P(x = f+g) = f$ where $x = f + g$ is a unique decomposition, then $P^2 = P$ and $P(E) = F$.

Quotients: Let E be a vector space and $F \subset E$ a linear subspace. Then define $x \sim y$ if and only if $x - y \in F$, which is an equivalence relation. Then $E/\sim = \{x + F : x \in E\}$ is the quotient set and $q : E \to E/F$, $q(x) = x + F$ is the well-defined, onto quotient map.

0.2. Topological spaces compatible with linear structure

By compatibility, we mean that addition and scalar multiplication are continuous operations. Not all topologies are such topologies (e.g. $\mathbb{R}^2$ with the discrete metric. The scalar multiplication is not continuous).

A special such family is normed spaces.

Norm: $\|\cdot\| : E \to R$ such that

(N1) $\|x\| = 0 \leftrightarrow x = \theta$ and $\|x\| > 0 \leftrightarrow x \neq \theta$

(N2) $\|\lambda x\| = |\lambda|\|x\|$

(N3) $\|x + y\| \leq \|x\| + \|y\|$

Banach Space: A normed space that is complete. An arbitrary normed space need not be complete. For Banach spaces, Baire's Category Thm. holds, resulting in the following:

Open Mapping Thm: A continuous and onto linear map $T : E \to F$ for E, F Banach spaces is an open map (takes open sets to open sets.)

Banach's Isomorphism Thm: Any continuous (linear) bijection between Banach spaces is a homeomorphism (itself and its inverse are continuous).

Uniform Boundedness Principle: Let $\mathcal{T}$ be a family of linear continuous maps $T : E \to F$ between two Banach spaces. If,

$$\forall x \in E \ \sup_{T \in \mathcal{T}} \|Tx\| < \infty$$

then

$$\sup_{T\in\mathcal{T}}\|T\|_{op} < \infty$$

Closed Graph Thm: Any linear continuous $T : E \to F$ between two Banach spaces having a closed graph is continuous.
$Gr(T) = \{(x, Tx) : x \in E\} \subset E \times F$ is a closed set if $(x_n, Tx_n) \to (x, y)$ then $x, y \in \mathrm{Gr}(T)$, i.e. $y = Tx$.

Schamder Basis: A sequence $\{x_n\}$ in a normed space E is a Schamder basis if

$$\forall x \in E\ \exists\lambda_n\text{'s such that } x = \sum_{i=1}^{\infty}\lambda_n x_n$$

Every normed space does not have a Schamder basis. If a normed space has a Schamder basis, then it is separable.

Topological Sum & Complementation: For a normed space, we say $E = F \oplus G$ is topological if it is a direct sum (algebraic) and the projection P is continuous.
The followings are equivalent for a normed space $E = F \oplus G$ where the sum is algebraic:

a) The sum is topological.
b) The map $T : F \times G \to E, T(f, g) = f + g$ is a homeomorphism.

Continuous Projection: If $P : E \to F$ is a continuous projection, then $F = \ker(I - P)$ is closed. In a topological space E with $F \subset E$ a linear subspace,

$$E/F = \{x + F : x \in E\} = E/\sim \quad \text{for} \quad x \sim y \leftrightarrow x - y \in F$$

where $E = F \oplus E/F$. Moreover, if E is a normed space (or a topological space in general), we require the projection to be continuous as well. In this case,

- The algebraic direct sum is topological.
- $\|x + F\|_{E/F} = \inf\{\|x - y\|_E : y \in F\}$ is a valid norm if and only if F is closed.
- f E is Banach, so is E/F.
- $E = F \oplus G$ is a topological sum if and only if $G \cong E/F$ topologically.

1. Inner Product & Hilbert Spaces

We will take the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ by convention.

1.1. Inner Product

Inner Product (Pre-Hilbert) Space: A lin space E is an inner product(pre-Hilbert) space is there exists a function $(\cdot|\cdot) : E \times E \to K$ satisfying

(P1) $(x|y) = \overline{(y|x)}$

(P2) $(x + y|z) = (x|z) + (y|z)$

(P3) $(\lambda x|y) = \lambda(x|y)$

P2 and P3 say that we have linearity in the first variable.

(P4) $(x|x) > 0$ if $x \neq 0$

One also has the following:

a) $(x|y + z) = (x|y) + (x|z)$

b) $(x|\lambda y) = \overline{\lambda}\,(x|y)$

Inner product is conjugate linear in the second variable.

c) $\forall x, y \;\; (\theta|x) = (x|\theta) = 0$

d) If $(x|z) = (y|z)$ for all z, then $x = y$. In particular if $(x|z) = 0$ for all z, then $x = \theta$

Induced Norm: The inner product induces the following norm: $\|x\| = (x|x)^{1/2}$. Then the following hold:

Parallelogram Law: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Polarization Identity: $4\,(x|y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - \|x - iy\|^2$
If $\mathbb{K} = \mathbb{R}$, we don't have the i terms.

Cauchy-Schwartz Inequality: $|(x|y)| \leq \|x\|\|y\|$

Every inner product space E is a normed space under the induced norm, and the inner product is continuous on $E \times E$. Moreover, if the norm of a normed space satisfies the parallelogram law, then it comes from an inner product.

1.2. Orthogonality

Orthogonality: If $(x|y) = 0$, then we say $x \perp y$.
A set $S \subset E$ is orthogonal to some x if for all $y \in S$, $x \perp y$.
For $S \subset E$, S itself is an orthogonal set if for all $x \neq y$, $x \perp y$.

Pythagorean Thm: For some orthogonal sequence $\{x_1, ..., x_n\}$,

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n}\|x_k\|^2$$

Any orthogonal set is linearly independent.

Orthonormality: A set S is orthonormal if it is orthogonal and for all $x \in S$, $\|x\| = 1$.

Gramm-Schmidt Orthonormalization: Let $\{x_n\}$ be a linearly independent sequence in an (infinite dimensional) inner product space. Then there is an *orthonormal* sequence $\{u_n\}$ such that

$$\langle\{x_1, ..., x_n\}\rangle = \langle\{u_1, ..., u_n\}\rangle$$

for all n. If $\{x_n\}$ is not linearly independent but $\langle\{x_1, ..., x_n\}\rangle$ is infinite dimensional, then there exists an orthonormal sequence $\{u_n\}$ and a k_n increasing sequence such that

$$\langle\{x_{k_1}, ..., x_{k_n}\}\rangle = \langle\{u_1, ..., u_n\}\rangle$$

Bessel's (In)Equality: Let $x_1, ..., x_n$ be orthonormal. Then for all x, for all n we have

$$\left\| x - \sum_{k=1}^{n} (x|x_k)\, x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x|x_k)|^2$$

and thus

$$\sum_{k=1}^{n} |(x|x_k)|^2 \leq \|x\|^2$$

meaning that the series $\sum_{k=1}^{n} |(x|x_k)|^2$ is absolutely convergent.

Hilbert Space: An inner product space that is complete under the induced norm is called a Hilbert Space.
Completion of an inner product space is again an inner product space, and hence a Hilbert space.

Convergence of an orthonormal sequence: Let $\{x_n\}$ be orthonormal in a Hilbert space. Then

$$\sum_{k=1}^{\infty} \lambda_k x_k \text{ convergent} \iff \sum_{k=1}^{\infty} |\lambda_k| < \infty$$

Total Sets: Let E be an inner product space. The $\{x_n\}$ is said to be total if

$$\forall x \in E \; (\forall n \;\; (x|x_n) = 0 \rightarrow x = \theta)$$

If $\{e_n\}$ is a (Schamder) basis in an inner product space, then it is linearly independent, total and its span is dense. (In fact, finite rational linear combinations of $\{e_n\}$ is dense.)

Basis, Totality, Parseval: Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space. Then the followings are equivalent:

a) $\{x_n\}$ is an (orthonormal) basis.
b) $\{x_n\}$ is total.
c) **Parseval's Identity:** $\sum_{n=1}^{\infty} |(x|x_n)|^2 = \|x\|^2$ for all x.

Basis, Separability, Totality: Let H be a Hilbert space. Then the following are equivalent:

a) H is separable.
b) H has an orthonormal basis.
c) H has a total sequence.

1.3. Isomorphisms

Homeomorphisms: Let X, Y be topological spaces. A map $T : X \to Y$ is said to be a homeomorphism if T is invertible and both T and T^{-1} are continuous.

Isomorphism: If further X, Y have linear structure (such as in normed spaces), then we require T be linear as well, in which case it is called an isomorphism.

Norm-isometry: If T also preserves (only) norms, i.e. $\|Tx\| = \|x\| \forall x$ then it is called a norm-isometry. An isometry may fail to be onto, even though it is one-to-one, invertible from its image with its inverse continuous on its image.

Isometric Isomorphism: It T is onto as well, then it is called an isometric isomorphism.

Hilbert space Isometry: If X, Y are Hilbert spaces, a linear map $T : X \to Y$ is a Hilbert space isometry if $(Tx|Ty) = (x|y)$. If it is also onto, it will be a Hilbert space isometric isomorphism.

Separability & Isomorphism Thm: Let H be a separable Hilbert space with field $\mathbb{K}$. Then

- If $n = \dim(H) < \infty$, i.e. H is finite dimensional, then $H \cong \mathbb{K}^n$
- If $\dim(H) = \infty$, i.e. H is infinite dimensional, then $H \cong l_2$

Here all isomorphisms are Hilbert space isomorphisms, i.e. $(Tx|Ty) = (x|y)$

1.4. Closed Linear Subspaces in a Hilbert Space

Annihilator: Let $S \subset H$ be a non-empty subset of a Hilbert space. The annihilator of S is defined as

$$S^{\perp} = \{x \in H : (x|s) \, \forall s \in S\}$$

Facts:

1. $S \subset (S^{\perp})^{\perp} = S^{\perp\perp}$
2. $S \subset P \to P^{\perp} \subset S^{\perp}$
3. $S^{\perp\perp\perp} = S^{\perp}$

For all non-empty $S \subset H$, $S^{\perp}$ is a linear closed subspace.

Unique Representations using Orthogonal Sets: Let $S, P \subset H$ be linear subspaces of a Hilbert space H. If $S \perp P$, then for all $x \in S + P = \{s + p : s \in S, \ p \in P\}$ where the sum is element-wise, there exists a unique representation $x = s + p$ where $s \in S, p \in P$. This means that $S + P = S \oplus P$ algebraically. If further S, P are closed, the $S + P = S \oplus P$ topologically and $S + P$ is closed.

Annihilators in Finite Dimension: Let $P \subset H$ be a finite dimensional subspace of a Hilbert space H. Then **facts**:

1. P is closed.
2. $H = P \oplus P^{\perp}$ topologically.
3. $P = P^{\perp\perp}$

1.5. Convex Sets and Minimizing Vector

Convexity: A segment joining x to y in a linear space is

$$\{z : z = \alpha x + (1-\alpha)y, 0 \leq \alpha \leq 1\}$$

If a set S contains the segment joining x to y for all $x, y \in S$, then it is said to be convex.
Facts:

1. Every linear subspace is convex.
2. In an inner product space, the line segment joining x to y can also be given as

$$\{z : \|x-y\| = \|x-z\| + \|z-y\|\}$$

   However, this is not true in a general normed space, certainly not in a metric space.
3. The unit ball, although not linear, is convex.

Minimizing Vector Thm: Let non-empty $S \subset H$ be a complete convex subset in an inner product space H. Then for all $x \in H$, there exists a unique $y_0 \in S$ such that $\|x - y_0\| \leq \|x - y\|$ for all $y \in S$. (i.e. $d(x, y_0) = d(x, S)$).
Corollaries:

1. If S is closed and convex in a Hilbert space H, then for all $x \in H$, there exists a unique $y_0 \in S$ such that $d(x, y_0) = d(x, S)$.
2. In every complete, convex subset S of an inner product space,there exists a unique $y_0 \in S$ of minimum norm (take $x = \theta$ in the theorem.)
   (One can take a closed and convex subset of a Hilbert space.)
3. In every closed linear subspace E of a Hilbert space H, for all $x \in H$ there exists a unique $y_0 \in E$ such that $d(x, y_0) = d(x, E)$.
4. Let S be a complete linear subspace of an inner product space H (or a closed linear subspace of a Hilbert space). Then for all $x \in H$, the minimizing vector (which exists by the thm.) satisfies $x - y_0 \perp S$ ($x - y_0 \in S^\perp$).

1.6. Orthogonal Complements & Projections

Orthogonal Complement & Projection: Let E be an inner product space and $F \subset E$ be a linear subspace. If $E = F \oplus G$ algebraically and $F \perp G$, then G is called an orthogonal complement to F and the projection $P : E \to F (= P(E))$ an orthogonal projection.
Facts on (algebraic) projections $P^2 = P$: Let $E = F \oplus G$, $P : E \to F$.

1. $Px = x \leftrightarrow x \in F$
2. $E = \text{Ran}(P) \oplus \ker(P)$
3. If E is a normed space and P is continuous, then $\text{Ran}(P)$ and $\ker(P)$ are closed linear spaces
4. If E is Banach, then the converse is also true. Namely, if F, G are closed linear subspaces and $E = F \oplus G$ algebraically, then P is continuous, i.e. the sum is topological.
5. (NEW) If E is an inner product space and P is an orthogonal projection, then
   a. $(x|Py) = (Px|y)$ for all $x, y \in E$.
   b. $(x|Px) = \|Px\|^2$, so $\|Px\| \leq \|x\|$, and so P is automatically continuous.

Hilbert Spaces & Orthogonal Complementation: Let H be a Hilbert space, $S \subseteq H$ be a closed linear subspace. Then

a. $H = S \oplus S^{\perp}$ topologically.
b. $S = S^{\perp\perp}$

therefore **facts:**

1. In a Hilbert space, every closed linear subspace is topologically (and in particular orthogonally) complemented, and $S = S^{\perp\perp}$.
2. The converse statement is also true: If E is a Banach space such that every closed linear subspace has a topological complement, then E is a Hilbert space (Lindenstrauss-Tzafriri 1971).
3. Let H be a Hilbert space and $F \subseteq H$. Then
   a. $\overline{F}^{\perp} = F^{\perp}$, $\overline{F} = F^{\perp\perp}$
   b. F is dense if and only if $F^{\perp} = \{\theta\}$
4. Let H be a Hilbert space and $S \subset H$. Then $S^{\perp\perp}$ is the smallest closed linear subspace of H containing S, i.e. $\overline{\langle S \rangle} = S^{\perp\perp}$. If further S is linear, then $\overline{S} = S^{\perp\perp}$.

1.7. Intermission: Continuous Linear Maps and the Dual

Operator Norm: Let E, F be normed spaces. We write

$$\mathcal{L}(E,F) = \{T : E \to F : T \text{ is linear and continuous}\}$$

$\mathcal{L}(E,F)$ is a normed space with respect to the operator norm defined as

$$\|T\|_{op} = \sup_{\|x\| \le 1} \|Tx\|$$

Equivalently,

$$\|T\|_{op} = \inf\{M > 0 : \|Tx\| < M\|x\| \ \forall x \in E\}$$

meaning if $\|Tx\| < M\|x\|$, then $\|T\|_{op} < M$. Also

$$\|T\|_{op} = \sup\left\{ \frac{\|Av\|}{\|v\|} \ : \ v \neq \theta \text{ for } v \in V \right\}$$

If F is Banach, then so is $\mathcal{L}(E,F)$.

Dual Space: Let E be a normed space. Then the dual of E is defined as

$$E' = \{f : E \to \mathbb{K} : f \text{ is continuous and linear}\} = \mathcal{L}(E, \mathbb{K})$$

E' is always Banach. The second dual $E'' = (E')'$ is defined similarly.

Canonical Injection Map: Let $x \in E$. Define $j : E \to E''$ which maps x to a functional $j(x) \in E''$. Now note that as $j(x) \in E''$, its domain is the functionals defined on E,i.e. $E' = \mathcal{L}(E, \mathbb{K})$. So, define $j(x)$, as usual, by how it is evaluated in an element f in the domain E', as $j(x)f = f(x)$.
$j(x)$ is linear, well-defined and

$$|j(x)f| = |f(x)| \le \|f\|\|x\|$$

meaning that $\|j(x)\| \le \|x\|$, so j is continuous. Furthermore, it is an isometry: $\|j(x)\| = \|x\|$. But it is not always onto, so not an isometric isomorphism in general.

1.8. The Dual H' of a Hilbert Space

Dual of an Inner Product Space: Let H be an inner product space. Define $T : H \to H'$ by $(Ty)x \triangleq (x|y)$. [1] Then

0. T is well-defined, i.e. $Ty \in H'$.
1. T is not linear but 'conjugate linear', i.e. $T(\alpha x + y) = \overline{\alpha}Tx + Ty$.
2. T is a norm-isometry.
3. If T is surjective, then H is a Hilbert space.

Riesz-Frechet Thm: (Converse of the last item above) Let H be a Hilbert space. The for all $f \in H'$, there exists a unique $y \in H$ such that $f(x) = (x|y)$ for all $x \in H$.

Inner Product of the Dual Space: Let H be a Hilbert space and let $f, g \in H'$. Then by Riesz-Frechet Thm, we know that there are unique $y_f, y_g \in H$ such that $f(x) = (x|y_f)$ and $g(x) = (x|y_g)$. Then the operation

$$(f|g)_{H'} = (y_g|y_f)_H$$

is a valid inner product, and H' an inner product space.

H' to H: Let H be a Hilbert space. The map $T : H' \to H, Tf = y_f$ (which is defined by the Riesz-Frechet Thm) is a norm-isometric isomorphism, but T is not linear but conjugate linear.

Summary:

- T above is a norm-isometric isomorphism, $H \cong H'$.
- T is a conjugate linear Hilbert space isometry, $H \cong H'$.
- But applying T twice, we obtain $H \cong H''$, a linear Hilbert space isometric isomorphism.
- If $\mathbb{K} = \mathbb{R}$, then by itself T is a Hilbert space isometric isomorphism.

1.9. The Adjoint Map

Adjoint Map: Let $T : H \to K$ be a linear and continuous map, where H, K are inner product spaces. Then there exists a unique linear continuous map $T^* : K \to H$ such that

$$(Th|k) = (h|T^*k)$$

for all $h \in H$, $k \in K$. This T^* is called the adjoint map.

Properties of the Adjoint Map: Let $S, T : H \to H$ be continuous and linear maps where H is a Hilbert space. Then

1. $(S + T)^* = S^* + T^*$
2. $(\lambda T)^* = \overline{\lambda}T^*$
3. $(T^*y|x) = (y|Tx)$
4. $(T^*)^* = T$

[1] Here notice that T maps the element $y \in H$ to the functional $Ty \in H'$, whose evaluation at a point $x \in H$ is defined as $(x|y)$.

5. $\|T^*T\| = \|TT^*\| = \|T\|^2 = \|T^*\|^2$
6. $T^*T = 0 \leftrightarrow T \equiv 0$, i.e. the zero function
7. $(ST)^* = T^*S^*$

Perp and Adjoint: Let $T : H \to H$ be a linear and continuous map, H a Hilbert space. If $S \subset H$ and $T(S) \subset V \subset H$, then

$$T^*(V^\perp) \subset S^\perp$$

Consequence: Let $T : H \to K$ be a linear and continuous map between Hilbert spaces. Let $M \subset H$ and $N \subset K$ be linear and closed. Then

$$T(M) \subset N \leftrightarrow T^*(N^\perp) \subset M^\perp$$

Kernel, Perp and Adjoint: Let $T : H \to K$ be a linear and continuous map between Hilbert spaces. Then

- $\ker(T) = T^*(K)^\perp$
- $\ker(T)^\perp = \overline{T^*(K)}$
- $\ker(T^*) = T(H)^\perp$
- $\ker(T^*)^\perp = \overline{T(H)}$

Thus, in particular, $H = \ker(T) \oplus \overline{T^*(K)} = \ker(T) \oplus \ker(T)^\perp$ topologically.

2. Operators on Hilbert Spaces

2.1. Some Operator Types and Their Properties

Operator Types: Let $T \in \mathcal{L}(H, H)$, i.e. a linear and continuous map between Hilbert spaces. We say that T is

isometric if and only if $T^*T = I$,

unitary if and only if $T^*T = TT^* = I$, therefore $T^{-1} = T^*$,

self-adjoint (symmetric, hermitian) if and only if $T = T^*$,

a projection if and only if $T^2 = T$ and $T = T^*$ (if orthogonal, then $T = T^*$ and continuity are automatic), and

normal if and only if $T^*T = TT^*$.

Remarks:

1. A unitary operator is isometric and normal.
2. A projection is self-adjoint.
3. Any self-adjoint operator is normal.
4. An operator is isometric and normal if and only if it is unitary.
5. For all T, TT^* and T^*T are always self-adjoint.

Operators being Zero: Let $T \in \mathcal{L}(H, H)$ where H is a Hilbert space. Then

1. If $S \subset H$ is a total set and $T(S) = \{\theta\}$, then $T \equiv 0$.
2. For $\mathbb{K} = \mathbb{C}$, if $(Tx|x) = 0$ for all $x \in H$, then $T \equiv 0$.
   For $\mathbb{K} = \mathbb{R}$, we further need T be self-adjoint.

Operators being Equal: (Corollary to the above) Let $S, T \in \mathcal{L}(H, H)$. Then

1. If $Tx = Sx$ on a total subset of H, then $T \equiv S$.
2. For $\mathbb{K} = \mathbb{C}$, if $(Tx|x) = (Sx|x)$ for all $x \in H$, then $T \equiv S$.

2.1.1. Isometric Operators

TFAE: Let $T \in \mathcal{L}(H, H)$, where H is a Hilbert space. Then, the followings are equivalent:

a) $\|Tx\| = \|x\|$ for all $x \in H$ (norm isometry)
b) $d(Tx, Ty) = d(x, y)$ for all $x, y \in H$ (metric isometry)
c) $(Tx|Ty) = (x|y)$ for all $x, y \in H$ (Hilbert space isometry)
d) T is an isometric operator, i.e. $T^*T = I$.

Not-Ontoness: An isometric operator NEED NOT be onto.

Closed Range: If $T \in \mathcal{L}(H, H)$ where H is Hilbert, is an isometry, then its range is closed.

2.1.2. Unitary Operators

TFAE: Let $T \in \mathcal{L}(H, H)$, where H is a Hilbert space. Then, the followings are equivalent:

a) T is unitary.
b) T^* is unitary.
c) T and T^* are both isometric.
d) T is isometric and T^* is injective.
e) T is isometric and onto.
f) $T^{-1} = T^*$.

Unitariness in ℓ_2: Let $T \in \ell_2 \to \ell_2$ be a diagonal operator, i.e. $Te_n = \mu_n e_n$ for all n where $\{e_n\}$ is the Kroenecker-Delta sequence. Then T is unitary if and only if $|\mu_n| = 1$ for all n.

2.1.3. Self-Adjoint Operators

TFAE: Let $T \in \mathcal{L}(H, H)$, where H is a Hilbert space, and $\mathbb{K} = \mathbb{C}$. Then the followings are equivalent:

a) T is self-adjoint.
b) $(Tx|y) = (x|Ty)$ for all $x, y \in H$.
c) $(Tx|x) = (x|Tx)$ for all $x \in H$.
d) $(Tx|x) \in \mathbb{R}$ for all $x \in H$.

Properties: Let $S, T \in \mathcal{L}(H, H)$ be self-adjoint, $R \in \mathcal{L}(H, H)$. Then

a) $S + T$ and λT for $\lambda \in \mathbb{R}$ are self-adjoint.
b) R^*R and RR^* are always self-adjoint.
c) ST is self-adjoint if and only if $ST = TS$, i.e. they commute.
d) R has a decomposition $R = R_1 + iR_2$ where both

$$R_1 = \frac{1}{2}(R + R^*) \qquad R_2 = \frac{1}{2i}(R - R^*)$$

are self-adjoint.
e) $\|T\|_{op} = \sup\{|(Tx|x)| : \|x\| \leq 1\} = \sup\{|(Tx|x)| : \|x\| = 1\}$

2.1.4. Normal Operators

TFAE: Let $T \in \mathcal{L}(H, H)$, where H is a Hilbert space, and $\mathbb{K} = \mathbb{C}$. Then the followings are equivalent:

a) T is normal, i.e. $TT^* = T^*T$.
b) T^* is normal.
c) $\|Tx\| = \|T^*x\|$ for all $x \in H$.

2.2. The Spectrum and Resolvent Sets

2.2.1. Inverse Operator

Invertibility: Let E, F be normed spaces. Then we say that $T \in \mathcal{L}(E, F)$ is invertible if there exists $S \in \mathcal{L}(F, E)$ such that $TS = I_F$ and $ST = I_E$, and we write $S = T^{-1}$.
If $T : E \to F$ has a topological inverse, then it has an algebraic inverse, so it is a bijection. However, in general a continuous bijection T may fail to have a continuous inverse.

Finite Dimension: Let E be a finite dimensional normed space, and $T \in \mathcal{L}(E, E)$. Then the followings are equivalent:

1. T is injective.
2. T is surjective.
3. T has a right inverse, i.e. $TS = I_E$.
4. T has a left inverse, i.e. $ST = I_E$.
5. T is invertible.

2.2.2. The Spectrum and Resolvent Sets

We assume $\mathbb{K} = \mathbb{C}$.

Spectrum & Resolvent Set: Let $T \in \mathcal{L}(H, H)$, where H is a Banach space. The set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$$

is called the spectrum of T. The set

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$$

is called the resolvent set of T.

Components of $\boldsymbol{\sigma(T)}$: If $\dim H = \infty$, there are three reasons as to why $(\lambda I - T)^{-1}$ may fail to exist:

- If $(\lambda I - T)$ is not **injective**, then $\lambda \in \sigma_p(T)$, the point spectrum. In this case λ is called an eigenvalue.
- If $(\lambda I - T)$ is injective but not surjective, although $(\lambda I - T)H \subset H$ is **dense**, then $\lambda \in \sigma_c(T)$, the continuous spectrum.
- If $(\lambda I - T)$ is not injective nor surjective, with not even $(\lambda I - T)H \subset H$ being dense, then $\lambda \in \sigma_r(T)$, the residual spectrum.

Then we have

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

and all components are disjoint.

Taylor-like Inversion: Let $A \in \mathcal{L}(E, E)$, where E is Banach and $\|A\|_{op} \leq 1$. Then $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$$

Sets of Invertible/Non-invertible Operators: Let E be a Banach space. Call the set of invertible operators as $G \subset \mathcal{L}(E, E)$. Then G is an open set, which makes the set of non-invertible operators $\mathcal{L}(E, E) \backslash G$ a closed set.

Compactness of $\boldsymbol{\sigma(T)}$: Let E be a Banach space and $T \in \mathcal{L}(E, E)$. Then $\sigma(T)$ is a compact subset of $\mathbb{C}$, and it is contained in the closed disc $\{z : |z| \leq \|T\|_{op}\}$.

2.2.3. Eigenvalues

Eigenvalue and Eigenvector: Let $T \in \mathcal{L}(E, E)$. Then $\lambda \in \mathbb{C}$ is called an eigenvalue if there exists some $x \neq \theta$ such that $Tx = \lambda x$, where x is called an eigenvector. Equivalently, $\lambda \in \mathbb{C}$ is an eigenvalue if and only if $\ker(\lambda I - T) \neq \{\theta\}$, and so ever $\theta \neq x \in \ker(\lambda I - T)$ is an eigenvector of λ.
Every eigenvector determines a unique eigenvalue, but an eigenvalue may have many eigenvectors. Furthermore, the eigenvectors of a given eigenvalue form a linear space.
$\lambda \in \mathbb{C}$ is an eigenvalue if and only if $(\lambda I - T)$ is nor injective and any $\theta \neq x \in \ker(\lambda I - T)$ is an eigenvector.
Some easy **examples**:

1) For μI, μ is the only eigenvalue.
2) Eigenvalues of a projection $P \neq I$ are exactly $\{0, 1\}$.
3) T is not injective if and only if $0 \in \sigma(T)$.

Finite Dimensional $\boldsymbol{H}$: If $\dim H < \infty$, then $\sigma(T) = \sigma_p(T)$, so every element of the spectrum is an eigenvalue.

Properties: Let $\lambda \in \mathbb{C}$ be an eigenvalue of $T \in \mathcal{L}(H, H)$, where H is a Hilbert space. Then the followings hold:

a) $|\lambda| \leq \|T\|_{op}$, which is true for any Banach space (and not just Hilbert spaces).
b) If T is self-adjoint, then $\lambda \in \mathbb{R}$ even if $\mathbb{K} = \mathbb{C}$.
c) If T is an isometry, then $|\lambda| = 1$.
d) If T is normal, i.e. $T^*T = TT^*$, then
   - d.i) x is an eigenvector of T if and only if x is an eigenvector of T^*.
   - d.ii) λ is an eigenvector of T if and only if $\overline{\lambda}$ is an eigenvector of T^*.
   - d.iii) If $\lambda \neq \mu$ are two eigenvalues of T, then their associated eigenvalue spaces are orthogonal.

2.3. Compact Operators

Compact Operator: Let $T \in \mathcal{L}(E, F)$, where E, F are normed spaces. T is called compact if $T(U_E)$, where U_E is the closed unit ball of E, is relatively compact, i.e. $\overline{T(U_E)} \subset F$ is compact. In other words, T takes a sequence in U_E to a sequence in F which has a convergent subsequence. The set of compact operators is denoted by $\mathcal{K}(E, F) \subset \mathcal{L}(E, F)$.
A couple of **remarks:**

- Any continuous finite dimensional map, i.e. $\dim(\mathrm{Ran}(T)) < \infty$, is compact.
- $I : E \to E$, where E is a normed space, is compact if and only if $\dim E < \infty$.

- If P is a continuous projection, then P is compact if and only if $P(E)$ is finite dimensional.
- Define $T : \ell_2 \to \ell_2$ as element-wise multiplication with fixed and bounded $\{\eta_n\}$, so $T\{x_n\} = \{\eta_n x_n\}$. If this T is compact, then $\eta_n \to 0$.
- If $T \in \mathcal{L}(E, F)$ and F is Banach, then T is compact if and only if $T(U_E)$ is precompact in F, i.e. its closure is compact.

Facts on Compact Operators: Let E, F be normed spaces.

a) $\mathcal{K}(E, F) \subset \mathcal{L}(E, F)$, meaning every compact operator is continuous.

b) $\mathcal{K}(E, F)$ is a closed linear subspace of $\mathcal{L}(E, F)$, provided that F is Banach.

c) If E_0, F_0 are normed spaces, $R \in \mathcal{L}(E_0, E)$, $S \in \mathcal{L}(F, F_0)$ and $T \in \mathcal{K}(E, F)$, then STR is compact.
This results says that $\mathcal{K}(E, E)$ is an ideal in the ring with identity over $\mathcal{L}(E, E)$.

d) Let H be a Hilbert space. Then $T \in \mathcal{L}(H, H)$ is compact if and only if $T^* \in \mathcal{L}(H, H)$ is compact.

Induced Normal Operators: Let $\{\lambda_k\}$ be a bounded sequence of complex numbers and H be a separable Hilbert space, with an orthonormal basis $\{x_n\}$. Then there exists a unique normal operator $T \in \mathcal{L}(H, H)$ such that it has eigenvalues $\{\lambda_k\}$ with their corresponding eigenvectors $\{x_k\}$, i.e. $Tx_k = \lambda_k x_k$ for all k. This T is given by

$$Tx = T\left(\sum_{k=0}^{\infty} \mu_k x_k\right) = \sum_{k=0}^{\infty} \lambda_k \mu_k x_k$$

with

$$\|T\|_{op} = \sup_k |\lambda_k|$$

If further we have $\lambda_k \to 0$, this normal T is also compact.

Finite Dimensional Kernel: Let $T \in \mathcal{K}(H, H)$, where H is a Hilbert space. Then for all $\lambda \neq 0$,

$$\dim(\ker(\lambda I - T)) < \infty$$

This theorem has the following consequence for $T \in \mathcal{K}(H, H)$: For $\lambda \neq 0$, $(\lambda I - T)H$ is closed in H.

Injective $\iff$ Surjective: Let $T \in \mathcal{K}(E, E)$ for a Banach space E. Then $\lambda I - T$ is injective if and only if it is surjective.

Spectrum of a Compact Operator: Let E be a Banach space and $T \in \mathcal{K}(E, E)$. Then every element of $\sigma(T)$, with the possible exception of 0, is an eigenvalue.

Eigenvalues of a Compact Operator: Let E be a Banach space with $\dim E = \infty$ and $T \in \mathcal{K}(E, E)$. Then the eigenvalues of T are countable and form a null sequence (i.e. converging to 0).

Range of a Compact Operator: The range of a compact operator is separable.

At Least One Non-zero Eigenvalue: Let $T \in \mathcal{K}(H, H)$, where H is a Hilbert space, be a non-zero and self-adjoint operator. Then, it has at least one non-zero eigenvalue, which is one of the following:

$$M = \sup_{\|x\|=1} (Tx|x) \qquad m = \inf_{\|x\|=1} (Tx|x)$$

Spectral Decomposition of Compact Self-Adjoint Operators: Let $T \in \mathcal{K}(H, H)$, where H is a Hilbert space, be self-adjoint. Then there exists an orthonormal sequence $\{f_n\}$ such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \left(x|f_n\right) f_n$$

where $\{\lambda_n\}$ is the sequence of eigenvalues of T counting multiplicities.

Spectral Mapping Theorem: Let $T \in \mathcal{K}(H, H)$, where H is a Hilbert space, be self-adjoint. For any entire function $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ on $\mathbb{C}$, one has

$$\sigma(f(T)) = f(\sigma(T))$$

where $f(T)$ is interpreted as

$$f(T) = \sum_{k=0}^{\infty} a_k T^k$$

2.4. Positive Operators

Take $\mathbb{K} = \mathbb{C}$.

Positive Operator: Let $T \in \mathcal{L}(H, H)$, where H is a Hilbert space. T is a positive operator if for all $x \in H$ we have $(Tx|x) \geq 0$. Therefore we say $S \leq T$ if and only if $((T - S)x|x) \geq 0$ for all $x \in H$, i.e. $T - S \geq 0$.

Positive Operators in $\mathbb{R}^{\boldsymbol{N}}$ with $\mathbb{K} = \mathbb{R}$: In this case the term "positive operator" corresponds to positive semi-definite (square) matrices, in which case we have all eigenvalues positive. The converse is also true if the matrix is symmetric (self-adjoint), i.e. symmetric (self-adjoint) operators are positive. However, not all positive operators are self-adjoint.

Positive Operators in $\mathbb{C}^{\boldsymbol{N}}$ with $\mathbb{K} = \mathbb{C}$: In this case, an operator (matrix) is positive if and only if all its eigenvalues are positive.

Positive Operators in Arbitrary $\boldsymbol{H}$ with $\mathbb{K} = \mathbb{C}$: In this case, the followings hold:

1. An operator T is positive if and only if it is self-adjoint and $\sigma(T) \subset [0, \infty)$.
2. If $T \geq 0$, then $T^n \geq 0$ for all $n \in \mathbb{N}^+$.
3. If $S, T \geq 0$, both are self-adjoint and they commute, i.e. $ST = TS$, then $ST \geq 0$. Both of them being positive is not enough.
4. If $T \geq 0$, then $\|Tx\|^2 \leq \|T\|_{op} (Tx|x)$.
5. Let $T_n \in \mathcal{L}(H, H)$ be an operator sequence such that

$$0 \leq T_1 \leq T_2 \leq \cdots \leq I$$

   Then there exists some $T \in \mathcal{L}(H, H)$ such that for all $x \in H$, $\lim T_n x = Tx$

Existence of Square Root: Let $T \in \mathcal{L}(H, H)$ for a complex Hilbert space H, and $T \geq 0$. Then there exists a unique $S \in \mathcal{L}(H, H)$ such that $S \geq 0$ and $S^2 = T$. Moreover, there exist polynomials p_n such that $p_n(T)x \to Sx$ for every $x \in H$ whenever $\|T\|_{op} \leq 1$.
The existence of these polynomials guarantee that S and T commute.

2.5. Spectral Decomposition of Compact Operators

Spectral Decompotision of Compact Operators: Let $T \in \mathcal{K}(H,H)$, where H is a Hilbert space. Then there exist orthonormal sequences $\{e_n\}$, $\{f_n\}$ and a non-negative scalar sequence λ_n such that for all $x \in H$,

$$Tx = \sum_{n=0}^{\infty} \lambda_n \,(x|e_n)\, f_n$$

Density of Finite Dimensional Operators: Let $\mathcal{F}(H,H)$ denote the set of all linear and continuous operators with finite dimensional range, and recall that they are all compact. Then $\overline{\mathcal{F}(H,H)} = \mathcal{K}(H,H)$, meaning finite dimensional operators are dense in $\mathcal{K}(H,H)$.

2.6. Polar Decomposition

Partial Isometry: Let H be a Hilbert space. An operator $U \in \mathcal{L}(H,H)$ is called a partial isometry if for some linear subspace $M \subset H$, $U|_M$ is an isometry and $U|_{M^\perp} \equiv 0$.
If $M = H$, then U is an isometry, and hence every isometry is a partial isometry.

Polar Decomposition: Let H be a Hilbert space and $T \in \mathcal{L}(H,H)$. Then there exist unique $P, U \in \mathcal{L}(H,H)$ such that $P \geq 0$ and U is a partial isometry on $\operatorname{Ran} P$ such that $T = UP$ and $P = U^*T$. Moreover, if T is invertible, then so is P and U is unitary. Here, P is defined to be

$$P = |T| = \sqrt{T^*T}$$

3. Some Applications

3.1. Fourier Series

$\mathcal{C}[-\pi,\pi]$ space: Let function space $\mathcal{C}[a,b]$ (real or complex valued) in general can be used to represent all signals with period $b-a$. As a prototype, we choose $\mathcal{C}[-\pi,\pi]$, because sines, cosines and complex exponentials have period 2π. We also equip it with the usual L_2 inner product and its induced norm:

$$(f|g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt \qquad \|f\| = \left(\int_{-\pi}^{\pi} |f(t)|^2 dt\right)^{1/2}$$

Bases: In this space, the following sequences are orthonormal:

$$\Bigg\{\underbrace{\frac{1}{\sqrt{2\pi}}e^{int}}_{e_n(t)}\Bigg\}_{n=-\infty}^{\infty} \qquad \Bigg\{\underbrace{\frac{1}{\sqrt{2\pi}}}_{u_0(t)}\Bigg\} \cup \Bigg\{\underbrace{\frac{1}{\sqrt{\pi}}\cos(nt)}_{u_n(t)}\Bigg\}_{n=1}^{\infty} \cup \Bigg\{\underbrace{\frac{1}{\sqrt{\pi}}\sin(nt)}_{v_n(t)}\Bigg\}_{n=1}^{\infty}$$

These two sets of orthogonal sequences are actually equivalent. One uses $\{e_n(t)\}$ if $\mathbb{K}=\mathbb{C}$, the other one if $\mathbb{K}=\mathbb{R}$.
Using **Stone-Weierstrass Approximation Theorem**, or its version on trigonometric polynomials, we can show that these sets are indeed orthonormal bases in $\mathcal{C}[-\pi,\pi]$

Fourier Coefficients: Traditionally, one does not work with the actual coordinates in these orthonormal bases but on the following definitions, called the Fourier coefficients, which are the multipliers of sines and cosines directly:

$$\left.\begin{array}{ll} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt & \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt & \text{for } n>1 \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt & \text{for } n>1 \end{array}\right\} \Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt)$$

Completion to $L_2[-\pi,\pi]$: However, this inner product space is not complete. It can be completed to a Hilbert space: $\widehat{\mathcal{C}[-\pi,\pi]}$ is isometrically isomorphic to $L_2[-\pi,\pi]$, a *measure* space, with the following domain:

$$L_2[-\pi,\pi] = \{[f]_\sim \mid f:[-\pi,\pi]\to\mathbb{K}, \int_{-\pi}^{\pi} |f(t)|^2 dm(t) < \infty\}$$

Here we need a couple of clarifications:

The differential $dm(t)$: This differential notation in the integral means that the integral is a *Lebesgue integral*, which generalized Riemann integrals. The inner product and norm integrals are again adapted to be Lebesgue integrals as well:

$$(f|g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dm(t) \qquad \|f\| = \left(\int_{-\pi}^{\pi} |f(t)|^2 dm(t)\right)^{1/2}$$

The equivalence relation $\sim$: The Lebesgue integral disregards "small sets", such as countably infinite or finite sets in the domain of integration. Therefore when considering functions, we must not take them as they are (which are modifiable at countably many point, Lebesgue integral just "won't care "), but instead as *equivalence classes* with the following definition:

$$f \sim g \iff \int_{-\pi}^{\pi} |f(t) - g(t)| dm(t) = 0$$

Meaning if Lebesgue integral sees no difference, so won't we.

Convergence: We have that $\{e_n\}$ is an orthonormal basis for the separable Hilbert space $L_2[-\pi, \pi]$. This means that the partial sum sequences converge for each $f \in L_2[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \left| \sum_{k=-n}^{n} (f|e_k)\, e_k(t) - f(t) \right|^2 dm(t) \underset{n}{\to} 0$$

But this is convergence in L_2-norm, which does not give us *point-wise convergence.*

Jump Discontinuities: If $f(t)$ has a jump discontinuity at some x_0 and the left and right "derivatives" exits, then the Fourier series converges point-wisely to $\frac{f(x_0^+)+f(x_0^-)}{2}$, which is the average of the left and right limits.
In particular, if $f'(x_0)$ exists at some x_0, then the Fourier series converges point-wisely to $f(x_0)$.

Uniform Convergence and Fejér's Theorem: We may even want the partial sum sequence $\sum_{n=-\infty}^{\infty} (f|e_n)\, e_n(t)$ to converge uniformly. For uniform convergence, in one sense, we have
Fejér's Theorem:
Let $f \in L_2[-\pi, \pi]$ such that $\|f\|_\infty < \infty$. Then the partial sum sequence $\sum_{n=-\infty}^{\infty} (f|e_n)\, e_n(t)$ converges uniformly to $f(t)$ in Cesàro's sense, meaning the running average of the partial sum sequence converges uniformly:

$$\frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} (f|e_n)\, e_n(t) \xrightarrow[\text{uniformly}]{} f(t)$$

Summary:

1) For $f \in L_2[-\pi, \pi]$, then the Fourier partial sums sequence $\sum_{k=-n}^{n} (f|e_k)\, e_k$ converges to f in L_2-norm.
2) The running averages of the partial sum sequence converges to f uniformly, which implies point-wise convergence and under certain additional assumptions (such as absolute convergence) L_2 convergence.
3) Similar arguments lead to theorems on $L_2[a, b]$ and $L_2^{2\pi}(\mathbb{R})$, the set of 2π-periodic functions on all $\mathbb{R}$.
4) If f and f' are piece-wisely continuous, then the Fourier partial sums sequence converges to the average of left and right limits point-wisely, and in L_2 sense.
5) If f is continuous and f' is piece-wisely continuous, then the Fourier partial sums sequence converges to f uniformly.
6) If f is differentiable, then the Fourier partial sums sequence converges to f uniformly.

3.2. Hilbert-Schmidt Operators

Hilbert-Schmidt Operators: Let H be a *separable* Hilbert space. $T \in \mathcal{L}(H, H)$ is called a Hilbert-Schmidt operator if $\sum\limits_{n=1}^{\infty} \|Te_n\|^2 < \infty$ for some orthonormal basis $\{e_n\}$.
The choice of the orthonormal basis in the definition is not important, as the sum given is it can be shown that the sum is the same for any given basis.

Compactness: Hilbert-Schmidt operators are compact. Therefore the sum $\sum\limits_{n=1}^{\infty} \|Te_n\|^2$ is exactly equal to the sum $\sum\limits_{k=1}^{|\lambda_k|^2}$. This yields the

Alternative Definition: An operator $T \in \mathcal{L}(H, H)$ is a Hilbert-Schmidt operator if and only if the eigenvalue sequence counting multiplicities of $|T|$ is a member of ℓ_2.

Hilbert Space of HS Operators: From this definition, one can define an inner product on the Hilbert-Schmidt operators themselves, given by

$$(T|S) = \sum_{i=1}^{\infty} \lambda_i \overline{\mu_i}$$

where $\{\lambda_i\}$ and $\{\mu_i\}$ are the eigenvalue sequences of $|T|$ and $|S|$ counting multiplicities, respectively.

Schatten Classes: The class of Hilbert-Schmidt operators are denoted by $\mathcal{S}_2 \subset \mathcal{K}(H, H)$. We can further generalize this idea of treating operators as spaces themselves, by characterizing each operator by its eigenvalue sequence counting multiplicities: One says an operator T belongs to $\mathcal{S}_p$ if its absolute eigenvalue sequence counting multiplicities, $\{\lambda_i\}$, is in ℓ_p. These classes are called as Schatten classes. In this sense,

- $\mathcal{K}(H, H)$ can be identified with $\{\lambda_i\} \in c_0$, and
- $\mathcal{L}(H, H)$ can be identified with $\{\lambda_i\} \in \ell_\infty$.

3.2.1. An Application to Fredholm Integral Equations

Fredholm Integral Equations: There are three types of such equations, where we try to solve for $u(x)$ for $x \in [a, b]$:

$$\int_a^b k(x,t)u(t)dt = \mu u(x) \qquad \text{Homogenous Type}$$

$$\int_a^b k(x,t)u(t)dt = f(x) \qquad \text{Inhomogenous Type}$$

$$\int_a^b k(x,t)u(t)dt = \mu u(x) + f(x) \qquad \text{General Type}$$

One can represent these integral equations using an operator: Take $\mathbb{K} = \mathbb{R}$, and let $T \in \mathcal{L}(L_2[a, b], L_2[a, b])$ and

$$(Tu)(x) = \int_a^b k(x,t)u(t)dt$$

Then the above equations become

$$Tu = \mu u \qquad \text{Homogenous Type}$$
$$Tu = f \qquad \text{Inhomogenous Type}$$
$$Tu = \mu u + f \qquad \text{General Type}$$

Facts:

1. T is linear, and $k(x,t)$ is called "the kernel function of T".
2. If $\int_a^b \int_a^b |k(x,t)|^2 dt dx < \infty$, then T is a Hilbert-Schmidt operator.
3. T is self-adjoint if and only if $k(x,t) = k(t,x)$.

Path to solution: Solving homogenous and inhomogenous Fredholm integral equations reduces to finding the eigenvectors of T, or finding $\ker(I-T)$ (after normalization).

Fredholm Operator: The operator $I-T$, where T is such as above and compact, is called a Fredholm operator.

Fredholm Alternative Theorem: If $T \in \mathcal{K}(H,H)$, then exactly one of the followings hold:

(a) $(I-T)x = y$ has a unique solution x for each given y, and $(I-T)$ is invertible.

(b) $\ker(I-T) \neq \{\theta\}$, meaning $I-T$ is not injective.

Further, if we have $\|T\|_{op} < 1$, then certainly we are in case (a).

3.2.2. An Application to Numerical Analysis: Galerkin's Method

Problem Setting: Let $T \in \mathcal{K}(H,H)$, where H is a separable Hilbert space, say with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$[2]. We are trying to solve the equation $(I-T)x = y$ for x with a given y.

Approach: We solve the problem in an n-dimensional subspace of H, spanned by $G_n = \langle e_1, \dots, e_n \rangle$ (WLOG). So, if $P_n : H \to G_n$ is the projection onto G_n, we are now trying to solve

$$(I-T)\underbrace{P_n x}_{x_n} = P_n y$$

where $x_n = \sum_{k=1}^{n} c_k^{(n)} e_k$. In this case, the problem reduces to finding the coefficients $c_k^{(n)}$. Therefore, one needs to solve

$$c_j^{(n)} - \sum_{k=1}^{n} c_k^{(n)} (Te_k | e_j) = (y|e_j) \qquad \text{for } j = 1, 2, ..., n$$

So we have n equations in n unknowns $c_1^{(n)}, c_2^{(n)}, ..., c_n^{(n)}$.

Convergence: If T is self-adjoint and compact with $0 < mI \leq (I-T)$ for some $m \in \mathbb{R}^+$, then $x \to x_n$ in norm where x is the true solution, i.e. $\|x_n - x\| \underset{n}{\to} 0$. If $\|x\|_{op} < 1$, the order condition is automatically satisfied.

[2] H is not actually given to be infinite dimensional in the notes, but to me it made more sense to take it so.

A. Internet Look-ups

Baire's Theorem: If a space S is either a complete metric space or a locally compact T2-space, then the intersection of every countable collection of dense open subsets of S is necessarily dense in S. [Wolfram]

T2-space: A topological space fulfilling the T_2 axiom: Any two points have disjoint neighborhoods. [Wolfram]

Locally Compact: A topological space X is locally compact if every point has a neighborhood which is itself is contained in a compact set. [Wolfram]

Operator Norm: Let V, W be two normed spaces over the same field, and $A : V \to W$ a linear map. A is continuous if and only if for some $c \in \mathbb{K}$ we have

$$\forall v \in V \ \ \|Av\| \leq c\|v\|$$

that is to say that A is bounded. Then, the equivalent definitions of operator norm are given as follows:

$$\begin{aligned} \|A\|_{op} &= \inf\{c \geq 0 : \ \|Av\| \leq c\|v\| \text{ for all } v \in V\} \\ &= \sup\{\|Av\| : \ \|v\| \leq 1 \quad \text{for } v \in V\} \\ &= \sup\{\|Av\| : \ \|v\| < 1 \quad \text{for } v \in V\} \\ &= \sup\{\|Av\| : \ \|v\| = 1 \text{ or } 0 \ \text{ for } v \in V\} \\ &= \sup\{\|Av\| : \ \|v\| = 1 \quad \text{for } v \in V\} && \text{(if and only if } V \neq \{\theta\}) \\ &= \sup\left\{\frac{\|Av\|}{\|v\|} : v \neq \theta \quad \text{for } v \in V\right\} && \text{(if and only if } V \neq \{\theta\}) \end{aligned}$$

Properties:

1. Operator norm is a norm on the space of all bounded operators from V and W, meaning all the norm axioms hold.
2. $\|Av\| \leq \|A\|_{op}\|v\|$ for all $v \in V$.
3. For maps $A : V \to W$ and $B : W \to X$, we have $\|BA\|_{op} \leq \|B\|_{op}\|A\|_{op}$, where BA denotes composition.

[Wikipedia]