# MATH422 Cheat Sheet

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## Contents



# 1 Topology of Subsets of Euclidean Spaces

## 1.1 Open and Closed Subsets of Sets in $\mathbb{R}^n$

**Balls:**

**Open:** Let $p \in \mathbb{R}^n$ and $r > 0$. The open ball centered at $p$ with radius $r$ is

$$\mathrm{O}_r\left(p;\ \mathbb{R}^n\right) = \{x \in \mathbb{R}^n : \|x - p\| < r\}$$

**Relatively Open:** Let $p \in A \subset \mathbb{R}^n$ and $r > 0$. The open ball *in* $A$ of radius $r$ is

$$\mathrm{O}_r\left(p;\ A\right) = \{x \in A : \|x - p\| < r\} = \mathrm{O}_r\left(p;\ \mathbb{R}^n\right) \cap A$$

**Closed:** Let $p \in \mathbb{R}^n$ and $r > 0$. The closed ball centered at $p$ with radius $r$ is

$$\mathrm{C}_r\left(p;\ \mathbb{R}^n\right) = \{x \in \mathbb{R}^n : \|x - p\| \leq r\}$$

**Relatively Closed:** Let $p \in A \subset \mathbb{R}^n$ and $r > 0$. The closed ball *in* $A$ of radius $r$ is

$$\mathrm{C}_r\,(p;\ A) = \{x \in A : \|x - p\| \leq r\} = \overline{\mathrm{O}_r\,(p;\ A)}$$

**Open/Closed Set:** A set $A \subset B$ is said to be open in $B$ if for every $p \in A$, there exists some $\varepsilon_p > 0$ real such that $\mathrm{O}_p\,(\varepsilon_p;\ B) \subset A$. A set $A \subset B$ is said the be closed in $B$ if its complement in $B$ is open in $B$. Setting $B = \mathbb{R}^n$, we obtain the regular open and closed set definitions.
For closed and open sets in $\mathbb{R}^n$, we have

- $\varnothing$ and $\mathbb{R}^n$ are both open and closed.
- Finite intersection of open sets is open.
- Finite union of closed sets is closed.
- Arbitrary union of open sets is open.
- Arbitrary intersection of closed sets is closed.

**Relative Opennes/Closedness as Intersection:** Let $A \subset \mathbb{R}^n$. A subset $S \subset A$ is open/closed in $A$ if and only if there exists some open/closed subset $U \subset \mathbb{R}^n$ such that $S = U \cap A$.
If in particular $A$ is open, then $U \cap A$ is open, and so relative openness is the same as general openness.

**Open Neighbourhood:** If $p \in A$, then an open neighbourhood of $p$ in $A$ is an open subset of $A$ containing $p$.

**Producing Open Rectangles:** For $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$,

i) for $U \underset{\text{open}}{\subset} A$ and $V \underset{\text{open}}{\subset} B$, then $U \times V$ is an open subset of $A \times B$.
ii) If $W \subset A \times B$ is an open subset of $A \times B$, then for every $(p_1, p_2) \in W$ there exist reals $\varepsilon_A, \varepsilon_B > 0$ such that $\mathrm{O}_{\varepsilon_A}\,(p_1;\ A) \times \mathrm{O}_{\varepsilon_B}\,(p_2;\ B) \subset W$.

**Closure:** There exists a smallest closed subset containing a given subset. Let $D \subset A \subset \mathbb{R}^n$. The closure of $D$ in $A$ is defined to be the intersection of all closed subsets of $A$ containing $D$, denoted $\overline{D}$ or sometimes $\mathrm{Cl}_A(D)$.

**Open/Closed Maps:** Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ and $f : A \to B$. Then $f$ is said to be an open map if for every $U \underset{\text{open}}{\subset} A$, $f(U)$ is also open in $B$. Similarly, $f$ is said to be a closed map if for every $C \underset{\text{clsd}}{\subset} A$, $f(C)$ is also closed in $B$.

## 1.2 Continuous Maps

**Continuity:** Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, and $f : A \to B$ be a map. The map $f$ is continuous if

**with Open Preimage:** for every $U \underset{\text{open}}{\subset} B$, the inverse image $f^{-1}(U)$ is open in $A$:

$$U \underset{\text{open}}{\subset} B \ \Rightarrow\ f^{-1}(U) \underset{\text{open}}{\subset} A$$

**with $\boldsymbol{\varepsilon - \delta}$:** for every $p \in A$ and every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $x \in A$ with $\|x - p\| < \delta$ then $\|f(x) - f(p)\| < \varepsilon$. This means

$$f\,(\mathrm{O}_\delta\,(p;\ A)) \subset \mathrm{O}_\varepsilon\,(f(p);\ B)$$

**with Open Neighbourhoods:** for every $p \in A$ and every subset $U \underset{\text{open}}{\subset} B$ containing $f(p)$, there exists $V \underset{\text{open}}{\subset} A$ containing $p$ such that $f(V) \subset U$.

**Continuity of Restrictions:** Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ and $f : A \to B$. Suppose that $A = A_1 \cup A_2$ and $f|_{A_1}$ and $f|_{A_2}$ are both continuous. If $A_1$ and $A_2$ are both open or both closed in $A$, then $f$ is continuous.

**Pasting Lemma:** A corollary to this is that we can define an overall function $f : A \to B$ in terms of two functions , say $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$, defined on both open or both closed subsets $A = A_1 \cup A_2$; given that $f_1$ and $f_2$ agree on the intersection $A_1 \cap A_2$.

**Continuity of Component Functions:** Let $A, B_1, \ldots, B_k$ be subsets of a Euclidean space and $f : A \to B_1 \times \cdots \times B_k$ a function. Denote with $f_i : A \to B_i$ the component functions of $f$, which can also be written as $\pi_i \circ f$ where $\pi_i$ is the projection function onto $B_i$. Then $f$ is continuous if and only if all its $f_i$ component functions are continuous.

## 1.3 Connectedness

**(Dis)Connected Sets:** Let $A \subset \mathbb{R}^n$. $A$ is said to be disconnected if we can write $A = A_1 \cup A_2$ where $A_1 \cap A_2 = \varnothing$ with $A_1$ and $A_2$ both open in $A$. If $A$ is not disconnected, then we say it is connected.
Equivalently,

- $A$ is connected if $A$ cannot be expressed as a disjoint union of two of its closed subsets.
- $A$ is connected if the only subsets of $A$ that are both open and closed are $A$ and $\varnothing$.

**Component:** Let $A \subset \mathbb{R}^n$. Then $C \subset A$ is a component of $A$ if it is non-empty, connected and not a proper subset of a connected subset of $A$.

**Intervals:** Let $A \subset \mathbb{R}$. Then $A$ is connected if and only if it is an interval (in the most general sense).

**Continuous Image:** Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, and $f : A \to B$ a continuous map. If $A$ is connected, so is $f(A)$.
"Continuous image of a connected set is connected."
This fact can be used to prove the Intermediate Value Theorem.

**Path Connectedness:** Let $A \subset \mathbb{R}^n$, $x, y \in A$. A path from $x$ to $y$ is a continuous map $c : [0, 1] \to A$ such that $c(0) = x$ and $c(1) = y$. $A$ is said to be path connected if there exists a path between any $x, y \in A$.
A path connected set is connected, but the converse statement need not hold.

## 1.4 Compactness

**Cover, Subcover, Finite Cover, Open Cover:** Let $A \subset \mathbb{R}^n$. A cover $\mathcal{U}$ of $A$ is a collection of subsets of $A$ whose union gives all of $A$.
If $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of $A$ indexed by some index set $I$, then a subcover of $\mathcal{U}$ is a subcollection of $\mathcal{U}$ that is itself a cover of $A$, i.e. it is of the form $\{U_j\}_{j \in J}$ where $J \subset I$.
A finite cover is one that is constituted of finitely many sets.
An open cover of $A$ constitutes of open subsets of $A$.

**(Un)Bounded Sets:** Let $A \subset \mathbb{R}^n$. Then $A$ is bounded if there exists some non-negative real $r$ such that $A \subset \mathrm{O}_r\,(0_n;\ \mathbb{R}^n)$.

**Compact Sets:** Let $A \subset \mathbb{R}^n$. We say that $A$ is compact if every open cover of $A$ has a finite subcover.
The union of finitely many compact sets is compact.
A compact subset of a compact set is closed. A closed subset of a compact set is compact.
Compact sets are bounded.

**Heine-Borel Theorem:** A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed in $\mathbb{R}^n$ and bounded.
The proof of this theorem uses

- A closed interval in $\mathbb{R}$ is compact.
- The product of finitely many compact subsets of a Euclidean space is compact.

**Continuous Image:** Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, and $f : A \to B$ a continuous map. If $A$ is compact, so is $f(A)$.
"Continuous image of a compact set is compact."
This fact can be used to prove the Extreme Value Theorem.

### 1.5 Homeomorphisms & Quotient Maps

**Homeomorphism:** Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ and $f : A \to B$. $f$ is a homeomorphism (or a topological equivalence) if it is a continuous bijection with continuous inverse.
Note that $f^{-1}$ is continuous if and only if $f$ is an open map.

**Partition:** Let $A \subset \mathbb{R}^n$. A collection $\mathcal{P} = \{P_i\}_{i \in I}$ of subsets of $A$ is called a partition if $\bigcup\limits_{i \in I} P_i = A$ and $P_i \cap P_j = \varnothing$ whenever $i \neq j$.

**Quotient Map:** Let $A, B \subset \mathbb{R}^n$. A map $q : A \to B$ is called a quotient map if it is surjective and $U \underset{\text{open}}{\subset} B$ if and only if $q^{-1}(U) \underset{\text{open}}{\subset} A$.
Note that by the second condition, a quotient map is by definition, automatically continuous.

**Identification Space:** If $X \subset \mathbb{R}^n$ and $\mathcal{P} = \{P_i\}_{i \in I}$ is a partition of $X$, then $Y \subset \mathbb{R}^m$ is an identification space of $X$ and $\mathcal{P}$ if there is a quotient map $q : X \to Y$ such that $\{q^{-1}(y) : y \in Y\} = \mathcal{P}$.
The interpretation of this definition is that the quotient map given collapses each part of $X$ in $\mathcal{P}$ to a single point, and each point in $Y$ is mapped on by some part of $X$, almost as if

$$P_i \overset{bij}{\longmapsto} y \qquad \text{(but not under } q\text{!)}$$

**Continuity & Quotient Maps:** Let $X$, $Y$ and $Z$ be subsets of Euclidean spaces, $f : X \to Y$ a quotient map and $g : Y \to Z$. Then $g$ is continuous if and only if $g \circ f$ is continuous.

**Homeomorphic Identification Spaces:** Let $X \subset \mathbb{R}^n$ and $\mathcal{P}$ be a partition of $X$. If $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^k$ are both identification spaces of $X$ and $\mathcal{P}$, then we have $Y \approx Z$, meaning $Y$ and $Z$ are the same spaces up to homeomorphism.

**"Sewing" Sets Together:** Let $X, Y \subset \mathbb{R}^n$ be disjoint sets and $X' \subset X$, $Y' \subset Y$. If $h : X' \to Y'$ is a homeomorphism, then we can define a partition $\mathcal{P}(h)$ on $X \cup Y$, which consists of pairs $\{x, h(x)\}$ for all $x \in X'$ and $\{z\}$ for all $z \in (X \setminus X') \cup (Y \setminus Y')$. A set $W \subset \mathbb{R}^m$ is said to be the result of attaching $X$ and $Y$ through $h$, denoted $X \cup_h Y$, if $W$ is the identification space of $X \cup Y$ and $\mathcal{P}(h)$.

**Continuous Maps and Compactness:** Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ and $f : A \to B$ a continuous map. Then if $A$ is compact

i) then $f$ is a closed map.
ii) and $f$ is a surjection, then $f$ is a quotient map.
iii) and $f$ is a bijection, then $f$ is a homeomorphism.

# 2 Topological Surfaces

## 2.1 Arcs, Disks, and 1-Spheres

**Basic Definitions:** We will give some specific names to the following sets as follows:

$$D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\} \qquad \text{(Standard Unit Disk in } \mathbb{R}^2\text{)}$$
$$\operatorname{Int} D^2 = \{x \in \mathbb{R}^2 : \|x\| < 1\} \qquad \text{(Standard Open Disk in } \mathbb{R}^2\text{)}$$
$$S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\} = \partial D^2 \qquad \text{(Unit Circle in } \mathbb{R}^2\text{)}$$

Then

**Arc:** A subset of $\mathbb{R}^n$ that is homeomorphic to the interval $[-1, 1]$ is called an arc.

**Disk:** A subset of $\mathbb{R}^n$ that is homeomorphic to $D^2$ is called a disk.

**1-Sphere:** A subset of $\mathbb{R}^n$ that is homeomorphic to $S^1$ is called a 1-sphere.

**Invariance of Domain:** Let $U \subset \mathbb{R}^n$ be homeomorphic to $\mathbb{R}^n$. Then $U$ must be open in $\mathbb{R}^n$.

**Non-homeomorphic Dimensions:** For $n$ and $m$ distinct positive integers, $\mathbb{R}^n \not\approx \mathbb{R}^m$.

**Boundaries and Interiors of Disks:** Let $B \subset \mathbb{R}^n$ be a disk, and $h_1, h_2 : D^2 \to B$ be homeomorphisms. Then

$$h_1(S^1) = h_2(S^1) \qquad \text{and} \qquad h_1(\operatorname{Int} D^2) = h_2(\operatorname{Int} D^2)$$

and so, independent from the homeomorphism generating the disk, we can make the following definitions: For any homeomorphism $h : D^2 \to B$

**Interior of a Disk:** $\operatorname{Int} B = h(\operatorname{Int} D^2)$

**Boundary of a Disk:** $\partial B = h(\partial D^2) = h(S^1)$

**Boundaries and Interiors of Arcs:** A similar fact can be proven about arcs and their boundaries as well: Their interior and boundaries are independent from the homeomorphism generating them. So, for any homeomorphism $h : [-1, 1] \to A$

**Interior of an Arc:** $\operatorname{Int} A = h(\operatorname{Int}[-1, 1]) = h((-1, 1))$

**Boundary of an Arc:** $\partial A = h(\partial[-1, 1]) = h(\{-1, 1\})$

**Identity Outside of a Disk:** Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism. $h$ is called as the identity outside a disk if there is some disk $B \subset \mathbb{R}^2$ such that $h|_{\mathbb{R}^2 \setminus A} = \text{id}$.

**Schönflies Theorem:** Let $C \subset \mathbb{R}^2$ be a 1-sphere. Then there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H(S^1) = C$ and $H$ is the identity outside a disk.
**Corollary:**

i) **Jordan Curve Theorem:** The set $\mathbb{R}^2 \setminus C$ has precisely 2 components, one bounded and another unbounded.

ii) The union of $C$ and the bounded component of $\mathbb{R}^2 \setminus C$ is a disk, of which $C$ is the boundary.
This second fact does not hold in dimensions higher than 2, as seen by the counter example of the trefoil knot.

iii) Let $B_1, B_2 \subset \mathbb{R}^2$ be two disks. Then there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H(B_1) = B_2$ and $H$ is the identity outside a disk.

### 2.2 Surfaces in $\mathbb{R}^n$

**(Topological) Surface:** A subset $Q \subset \mathbb{R}^n$ is called a (topological) surface if each point $p \in Q$ has an open neighbourhood that is homeomorphic to the open unit dist $\operatorname{int} D^2$.

**"Sameness" of Surfaces:** Two surfaces $Q_1$ and $Q_2$ are considered to be the same if they are homeomorphic.

### 2.3 Surfaces via Gluing

**Polygonal Disk:** A polygonal disk is a disk that sits in some plane in $R^n$ whose boundary is a polygon.

**Gluing Scheme:** If $D$ is a polygonal disk, a gluing scheme $\mathcal{S}$ for the edges of $D$ is a labelling of each edge of $F$ with an arrow and a letter, where each used letter appears on precisely two edges.

**Affine Linear Map:** An affine linear map $L : [a, b] \to [c, d]$ is a map that satisfies

$$L\left((1-t)a + tb\right) = (1-t)L(a) + tL(b) \qquad \text{for } t \in [0,1]$$

**Edge Sets:** Let $D$ be a polygonal disk and $\mathcal{S}$ be a gluing scheme on the edges of $D$. Then, $\mathcal{S}$ partitions the edges of $D$ into 2-element sets where the edges matched under $\mathcal{S}$ are paired, say $E_1, \ldots, E_k$. These are called as the edge sets.

**Vertex Sets:** For each edge set, we have two affine linear maps that send one edge to the other such that the end points are matched according to the directions set by $\mathcal{S}$. Of course, one is the inverse of the other, so we can represent the mapping of each edge set with a single $L_i$ by choosing one of these mappings, and the other one is $L_i^{-1}$. Then define the following partition:

$$[x] = \{y \in D : y = L_i^{\pm 1} \circ L_j^{\pm 1}(x) \text{ for some } i, j \in \{0, 1, \ldots, k\}\}$$

where we write $L_0$ for the identity map. These sets, which form a partition of the vertices of $D$, are called the vertex sets, denoted $\mathcal{P}(\mathcal{S})$ or $\mathcal{P}(L_1, \ldots, L_k)$.

**Constructing Surfaces:** Let $D$ be a polygonal disk and $\mathcal{S}$ a gluing scheme on its edges. A subset $X \subset \mathbb{R}^n$ is said to be obtained from $D$ and $\mathcal{S}$ if $X$ is an identification space of $D$ and $\mathcal{P}(\mathcal{S})$, that is, there is a quotient map $q : D \xrightarrow{surj.} X$ such that if $x, y \in D$ are points, then $q(x) = q(y)$ if and only if $x$ and $y$ are in the same set in $\mathcal{P}(\mathcal{S})$.
For each polygonal disk $D$ and gluing scheme $\mathcal{S}$ on $D$, there is a surface $Q \subset \mathbb{R}^n$ that is obtained from $D$ and $\mathcal{S}$.

**Compact Connected Surfaces via Gluing:** Let $Q \subset \mathbb{R}^n$ be a compact connected surface. Then there is a polygonal disk $D$ and a gluing scheme $\mathcal{S}$ for the edges of $D$ such that $Q$ is obtained from $D$ and $\mathcal{S}$.

This fact is proved based on another statement asserting any surface can be triangulated. See Section 3 for more.

**Various Important Surfaces and Their Gluing Schemes:**

**Möbius Strip:** $M^2$. $M^2$ is not really a surface as it has a boundary, but it appears within many important surfaces, so it is mentioned here.

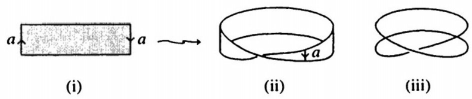


Figure 2.4.7

*E. Bloch*

**Sphere:** $S^2$, no need to elaborate.

**Torus:** $T^2$

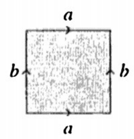


Figure 2.4.2


Figure 2.4.3

*E. Bloch*

**Klein Bottle:** $K^2$

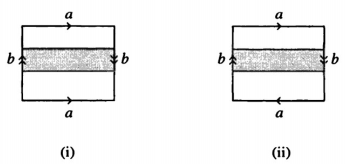


Figure 2.4.9

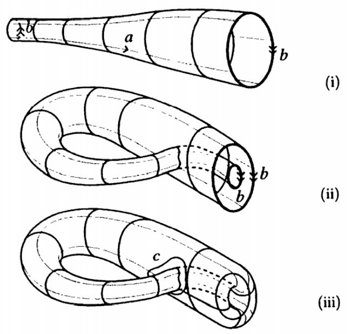


Figure 2.4.10

*E. Bloch*

$K^2$ can be obtained from two $M^2$'s via a homeomorphism of their boundaries.:

$$K^2 = (M^2)_1 \cup_h (M^2)_2$$

**Projective Plane:** $P^2$

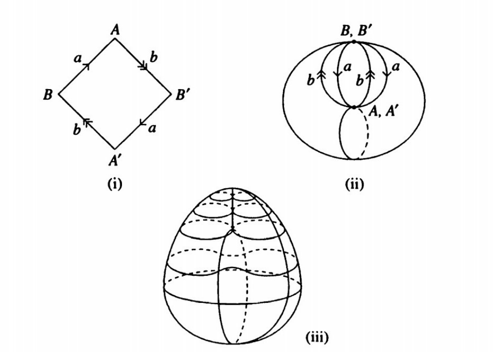

Figure 2.4.11

*E. Bloch*

The removal of an open disk from $P^2$ yields $M^2$:

$$P^2 \setminus \operatorname{int} B \approx M^2$$

This means that we can form a $P^2$ using an $M^2$ and a disk $D^2$ via a homeomorphism of their boundaries.

## 2.4 Properties of Surfaces

**Compactness:** Any surface $Q$ obtained from a polygonal disk by gluing is compact, as they are the continuous images (via the quotient map) of the compact $D^2$. So, any compact and connected surface is compact.

**Oritentability:** A surface is orientable if it does not contain a Möbius strip. It is non-orientable if it does.

**Connectedness:**

**Around Points:** Let $Q$ be a surface and $q \in Q$. Then $q$ has an open neighbourhood in $Q$ which is path connected.

**Overall:** A surface in $\mathbb{R}^n$ is connected if and only if it is path connected.

## 2.5 Connected Sum and Classification of Compact Connected Surfaces

**Connedted Sum:** Let $Q_1$ and $Q_2$ be compact and connected surfaces. For each, choose a disk $B_1$ and $B_2$, respectively. Let $h : \partial B_1 \to \partial B_2$ be a homeomorphism. The attaching space

$$(Q_1 \setminus \operatorname{int} B_1) \cup_h (Q_2 \setminus \operatorname{int} B_2)$$

is called the connected sum of $Q_1$ and $Q_2$, denoted $Q_1 \# Q_2$.

**Exitence and Uniqueness upto Homeomorphism:** This attaching space indeed exists and is a surface in some $\mathbb{R}^n$. Any two surfaces obtained in such a way are homeomorphic.

**Basic Properties:** Let $A$, $B$ and $C$ be compact and connected surfaces. Then

i) $A \# B \approx B \# A$
ii) $(A \# B) \# C \approx A \# (B \# C)$
iii) $A \# S^2 \approx A$

Notice that these properties yield a commutative group-like structure, where $S^2$ acts like the identity. We are only missing the inverses of each element. This algebraic structure is called a commutative semigroup. The next property illustrates that we indeed *cannot* have inverses, other than the trivial case.

**Trivial "Inverses":** Let $A$ and $B$ be compact connected surfaces. If we have $A\#B \approx S^2$, then $A \approx B \approx S^2$.

**Connected Sum of Important Surfaces:** The following homeomorphisms hold:

$P^2\#P^2 \approx K^2$**:** This one is easy to prove. Remember that we are removing the interior of a disk while constructing the connected sum. Then, the two $P^2$ after the removal of the said disk are homeomorphic to $M^2$, and then we are taking a homeomorphism of their boundary, which we had seen results in $K^2$.

$P^2\#T^2 \approx P^2\#P^2\#P^2$

**Orientability and Connected Sum:** Let $Q_1$ and $Q_2$ be compact connected surfaces. Then $Q_1\#Q_2$ is orientable if and only if both $Q_1$ and $Q_2$ are orientable.

**Clasification of Compact Connected Surfaces:** Any compact connected surface is homeomorphic to a sphere, a connected sum of tori or a connected sum of projective planes, that is, it belongs to the following list:

- $S^2$,
- $T^2$, $T^2\#T^2$, $T^2\#T^2\#T^2$, $\ldots$
- $P^2$, $P^2\#P^2$, $P^2\#P^2\#P^2$, $\ldots$

The surfaces in this list are all distinct (i.e. not homeomorphic to one another).

# 3 Simplicial Complexes

## 3.1 Affine Preliminaries

**Affine Combination, Span, Subspace & Independence:** Let $x_0, \ldots, x_k \in \mathbb{R}^n$. An affine combination of these points, with coefficients $t_0, \ldots, t_k \in \mathbb{R}$ is the linear combination

$$\sum_{i=0}^{k} t_i x_i \quad \text{with} \quad \sum_{i=0}^{k} t_i = 1$$

A set $X \subset \mathbb{R}^n$ is said to be an affine subspace if it is closed under affine combinations. The affine span of these points is the set of all affine combinations of these points:

$$\operatorname{aspan}(x_0, \ldots, x_k) = \left\{ y \in \mathbb{R}^n : y = \sum_{i=0}^{k} t_i x_i \text{ with } \sum_{i=0}^{k} t_i = 1 \right\}$$

The set $\{x_0, \ldots, x_k\}$ is said to be affinely independent if

$$\sum_{i=0}^{k} t_i x_i = 0_n \text{ with } \sum_{i=0}^{k} t_i = 0 \;\Rightarrow\; t_i = 0 \;\forall i = 0, \ldots, k$$

**Relation to Linear Independence:** The set $\{x_0, \ldots, x_k\}$ is affinely independent if and only if the set $\{x_1 - x_0, \ldots, x_k - x_0\}$ is linearly independent.

**Uniqueness:** The set $\{x_0, \ldots, x_k\}$ is affinely independent if and only if every $y$ in the affine span of $\{x_0, \ldots, x_k\}$ is uniquely expressible as an affine combination of $x_0, \ldots, x_k$. The coefficients $t_i$ in this affine combination are called the barycentric coordinates of $y$.

**Affine Linear Map:** Let $X \in \mathbb{R}^n$ be an affine subspace. A map $F : X \to \mathbb{R}^m$ is an affine linear map if it preserves affine combinations, that is if $x_0, \dots, x_k \in X$ and $t_0, \dots, t_k \in \mathbb{R}$ with $\sum_{i=0}^{k} t_i = 1$, then

$$F\left(\sum_{i=0}^{k} t_i x_i\right) = \sum_{i=0}^{k} t_i F(x_i)$$

## 3.2 Convex Sets & Simplices

**Convex Sets:** For $v, w \in \mathbb{R}^n$, let $\overline{vw}$ denote the line segment between $v$ and $w$, given by

$$\overline{vw} = \{x \in \mathbb{R}^n : x = tv + (1-t)w \text{ for } 0 \leq t \leq 1\}$$

A subset $X \subset \mathbb{R}^n$ is said to be convex if for every pair of points $v, w \in X$, the line segment $\overline{vw}$ is entirely contained in $X$.

**Image Under Affine Linear Maps:** For a convex set $C \subset \mathbb{R}^n$ and an affine linear map $F : \mathbb{R}^n \to \mathbb{R}^m$, the set $F(C)$ is also convex. This means that convexity is preserved under affine linear maps.

**Convex Hull:** It is intuitive (and true) that every set $X$ is contained in some convex set, so we might ask for the smallest such set. Let $X \subset \mathbb{R}^n$ be any set. The convex hull of $X$, denoted $\operatorname{conv} X$, is defined by

$$\operatorname{conv} X = \bigcap \{C \subset \mathbb{R}^n : X \subset C \text{ and } C \text{ is convex}\}$$

This set is indeed convex, and it a subset of every convex set containing $X$.

**Simplex:** Let $a_0, \dots, a_k \in \mathbb{R}^n$ be affinely independent points where $k$ is a non-negative integer.

**Via Convex Hull:** The simplex spanned by the points $a_0, \dots, a_k$ is the convex hull of these points, denoted

$$\langle a_0, \dots, a_k \rangle = \operatorname{conv}\{a_0, \dots, a_k\}$$

The points $a_0, \dots, a_k$ are called the vertices of the simplex, and $\langle a_0, \dots, a_k \rangle$ is said to be $k$-dimensional, and called a $k$-simplex. This makes sense, given the linear independence relation above. Our $k+1$ many points give us a $k$-dimensional object.


*Source*

**Via Affine Combinations:** A simplex is equivalently expressed in terms of its vertices' affine combinations with an extra condition. Let $a_0, \dots, a_k \in \mathbb{R}^n$ be affinely independent points, where $k \in \mathbb{N}^+$. Then

$$\langle a_0, \dots, a_k \rangle = \left\{ x \in \mathbb{R}^n : x = \sum_{i=0}^{k} t_i a_i \text{ with } \sum_{i=0}^{k} t_i = 1 \textbf{ AND } t_i \geq 0 \ \forall i \right\}$$

again, these $t_i$ coordinates are unique, and are called barycentric coordinates.

**Unique Vertices:** Let $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_p\}$ be two sets of affinely independent points in $\mathbb{R}^n$. Then we have

$$\langle a_0, \ldots, a_k\rangle = \langle b_0, \ldots, b_p\rangle \;\Rightarrow\; \{a_0, \ldots, a_k\} = \{b_0, \ldots, b_p\}$$

This means that the set of vertices of a simplex uniquely determines the simplex.

**Unique Affine Subspace:** Let $\mu = \langle a_0, \ldots, a_k\rangle$ be a k-simplex in $\mathbb{R}^n$. Then the affine subspace aspan $(a_0, \ldots, a_k)$ is the unique affine subspace ($k$-plane) containing $\mu$.

**Face of a Simplex:** Let $\sigma = \langle a_0, \ldots, a_k\rangle$ be a $k$-simplex in $\mathbb{R}^n$. A face of $\sigma$ is the simplex generated by a non-empty subset of $\{a_0, \ldots, a_k\}$. If this subset is proper subset, then the corresponding face is called a proper face. A face of $\sigma$ that is an $r$-simplex is called an $r$-face of $\sigma$.

**Combinatorial Boundary:** The combinatorial boundary of a $k$-simplex $\sigma = \langle a_0, \ldots, a_k\rangle$ is the union of all the proper faces of $\sigma$.

$$\operatorname{Bd}\sigma = \bigcup_{\tau \subsetneq \{a_0,\ldots,a_k\}} \langle\tau\rangle$$

Mind the abuse of notation.

The combinatorial boundary, by this logic, is given as

$$\operatorname{Bd}\langle a_0, \ldots, a_k\rangle = \Bigg\{ x \in \mathbb{R}^n : x = \sum_{i=0}^{k} t_i a_i \text{ with } \sum_{i=0}^{k} t_i = 1, \\ t_i \geq 0 \text{ for some } i \text{ and there exists } j \text{ such that } t_j = 0 \Bigg\}$$

Notice how imposing a zero in the barycentric coordinates forces the elements to be on a proper face. If there are $s$ zeros with $0 < s \leq k$ in the barycentric coordinates of some $x$, then $x$ lies on a $(k-s)$-face.

**Combinatorial Interior:** The combinatorial interior is the set given by

$$\operatorname{Int}\sigma = \sigma - \operatorname{Bd}\sigma$$

The combinatorial interior, by this logic, is given as

$$\operatorname{Int}\langle a_0, \ldots, a_k\rangle = \left\{ x \in \mathbb{R}^n : x = \sum_{i=0}^{k} t_i a_i \text{ with } \sum_{i=0}^{k} t_i = 1, t_i > 0 \;\forall i \right\}$$

Notice how the strict positivity constraint keeps the points strictly "inside" the simplex and not on the boundary.

**General Disks and Spheres:** We will use of the following notation from now on:

$$D^k = \{x \in \mathbb{R}^n : \|x\| \leq 1\} \qquad \text{Closed unit } k\text{-disk}$$
$$S^{k-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \qquad \text{Unit } (k-1)\text{-sphere}$$

We had already introduced $D^2$, $S^2$ and $S^1$ already.

**Simplices & Boundaries as Disks & Spheres:** Let $\sigma = \langle a_0, \ldots, a_k\rangle$ be a $k$-simplex in $\mathbb{R}^n$. Then, there is a homeomorphism

$$h : D^k \to \sigma$$

such that

$$h(S^{k-1}) = \operatorname{Bd}\sigma$$

If $\sigma$ is a 1-simplex, then it is an arc. If it is a 2-simplex, then it is a disk. In both cases, we have

$$\operatorname{Bd}\sigma = \partial\sigma \qquad \& \qquad \operatorname{Int}\sigma = \operatorname{int}\sigma$$

**Compactness and Connectedness:** Every $k$-simplex $\sigma = \langle a_0, \dots, a_k \rangle$ and $\mathrm{Bd}\,\sigma$ are compact and path connected.

### 3.3 Simplicial Complexes

**Simplicial Complex:** A simplicial complex $K$ in $\mathbb{R}^n$ is a finite collection of simplices in $\mathbb{R}^n$ such that

i) if $\sigma \in K$, then all faces of $\sigma$ are in $K$.
ii) if $\sigma, \tau \in K$ with non-empty intersection, then $\sigma \cap \tau$ must be a face of both $\sigma$ and $\tau$.

For a simplicial complex $K$, $K^{((i))}$ is defined as the set of all $i$-simplices in $K$.

**Subcomplex:** A subcollection $L$ of $K$ is a subcomplex of $K$ if $L$ itself is a simplicial complex.

**Star & Link:** Let $K$ be a simplicial complex in $\mathbb{R}^n$, and $\sigma \in K$. The star of $\sigma$ is defined as

$$\mathrm{star}\,(\sigma, K) = \{\eta \in K : \eta \text{ is a face of a simplex in } K\text{, which has } \sigma \text{ as a face}\}$$

Based on this definition, the link of $\sigma$ is defined as

$$\mathrm{link}\,(\sigma, K) = \{\eta \in \mathrm{star}\,(\sigma, K) : \eta \cap \sigma = \varnothing\}$$

**Simplicial Maps:** Let $K$ be a simplicial complex in $\mathbb{R}^n$ and let $L$ be a simplicial complex in $\mathbb{R}^m$. A map

$$f : K^{((0))} \to L^{((0))}$$

is called a simplicial map if

$$\langle a_0, \dots, a_i \rangle \in K \;\Rightarrow\; \langle f(a_0), \dots, f(a_i) \rangle \in L$$

**Simplicial Isomorphism:** Such a map is a simplicial isomorphism if it is bijective, and its inverse is also a simplicial map. If there is a simplicial isomorphism between $K$ and $L$, we say that they are simplicially isomorphic.

**Underlying Space:** Let $K$ be a simplicial complex in $\mathbb{R}^n$. The underlying space of $K$, denoted $|K|$, is the subset of $\mathbb{R}^n$ given by the union of all simplices in $K$.

$$|K| = \bigcup_{\sigma \in K} \sigma$$

Notice of $|K|$ is different from $K$ as a set: $K$ contains a finite amount of finite sets, while $|K|$ contains points from $\mathbb{R}^n$ and is most likely an uncountable set (it is countable if $K = K^{((0))}$)

**Unique Simplex per Point:** For each $x \in |K|$, there exists a unique simplex $\eta \in K$ such that $x \in \mathrm{Int}\,\eta$. Otherwise, two simplices intersect on a set that is not a face of a simplex in $K$.

**Subdivision:** Let $K$ and $K'$ be simplicial complexes in $\mathbb{R}^n$. The simplicial complex $K'$ is said to subdivide K if $|K| = |K'|$ and every simplex in $K'$ is a (not necessarily proper) subset of a simplex of $K$.

**Induced Map of a Simplicial Map:** Let $K$ be a simplicial complex in $\mathbb{R}^n$ and let $L$ be a simplicial complex in $\mathbb{R}^m$, with $f : K^{((0))} \to L^{((0))}$ a simplicial map. The induced map of the underlying spaces $|K|$ and $|L|$ is the map $|f| : |K| \to |L|$ defined by extending $f$ affinely over each simplex.

**Continuity:** The induced map $|f|$ of a simplicial map $f$ is continuous.

**Isomorphic Spaces:** If $f$ is a simplicial isomorphism between $K$ and $L$, then $|K| \approx |L|$.

**Isomorphic Subdivision Spaces:** If $K$ and $L$ have simplicially isomorphic subdivisions, then $|K| \approx |L|$.

**Connectedness:** Let $K$ be a simplicial complex in $\mathbb{R}^n$. Then, the following are equivalent:

i) $|K|$ is path connected.
ii) $|K|$ is connected.
iii) For any two simplices $\sigma$ and $\tau$ in $K$, there is a collection of simplices $\eta_i$ of $K$

$$\tau = \eta_1, \eta_2, \ldots, \eta_p = \sigma$$

such that $\eta_i \cap \eta_{i+1} \neq \varnothing$ for all $i = 1, \ldots, p-1$

**Simplicial Quotient Maps:** Let $K$ be a simplicial complex in $\mathbb{R}^n$ and $L$ a simplicial complex in $\mathbb{R}^m$. A simplicial map $f : K^{((0))} \to L^{((0))}$ is a simplicial quotient map if

i) For every simplex $\langle b_0, \ldots, b_p \rangle \in L$, there exists a simplex $\langle a_0, \ldots, a_p \rangle \in K$ such that $f(a_i) = b_i$.
ii) If $a, b \in K^{((0))}$ are two vertices of a common simplex, then $f(a) \neq f(b)$.

Notice how this definition is in parallel with the usage of quotient maps in the identification space definition.

**Induced Map:** The induced map $|f| : |K| \to |L|$ is a quotient map.

**Inverse Image under the Induced Map:** Let $y \in |l|$ be a point, and $\eta = \langle b_0, \ldots, b_k \rangle \in L$ be the unique simplex with $x \in \operatorname{Int} \eta$ with strictly positive barycentric coordinates $t_i$, i.e. $y = \sum\limits_{i=0}^{k} t_i b_i$, $\sum\limits_{i=0}^{k} t_i = 1$ and $t_i > 0$. Then, $|f|^{-1}(y)$ contains all points $x \in |K|$ with the same barycentric coordinates contained in some (possibly multiple) simplex $\langle a_0, \ldots, a_k \rangle \in K$ such that $f(a_i) = b_i$.

**Admissible Partition:** Let $K$ be a simplicial complex. An admissible partition of $K^{((0))}$ is a collection $\mathcal{V} = \{A_i\}_{i \in I}$ of disjoint subsets of $K^{((0))}$ such that

i) $\bigcup_{i \in I} A_i = K^{((0))}$
ii) No two vertices of the same simplex are in the same set $A_i$.

**Existence of an Image Complex:** For any admissible partition $\mathcal{V}$ of $K^{((0))}$ of a simplicial complex, there exists some simplicial complex $K'$ in some $\mathbb{R}^m$ and a simplicial quotient map $f : K^{((0))} \to K'^{((0))}$ such that

$$\left\{ f^{-1}(v) : v \in K'^{((0))} \right\} = \mathcal{V}$$

**Induced Partition:** An admissible partition of $\mathcal{V}$ of $K^{((0))}$ induces a partition of $|K|$, denoted $\mathcal{P}(\mathcal{V})$, as follows: Two points $x, y \in |K|$ are in the same partition if and only if for $x \in \operatorname{Int}\langle a_0, \ldots, a_k \rangle$ and $y \in \operatorname{Int}\langle b_0, \ldots, b_k \rangle$ of $K$ such that

i) the 0-simplices $a_i$ and $b_i$ are in the same set in the partition $\mathcal{V}$, and
ii) $F(x) = y$ under the unique affine linear map $F : \langle a_0, \ldots, a_k \rangle \to \langle b_0, \ldots, b_k \rangle$ with $F(a_i) = b_i$ (meaning they have the same barycentric coordinates in their own $k$-simplices).

**Induced Preimage of the Image Subspace:** We know that an image simplicial complex $K'$ exists for any admissible partition $\mathcal{V}$ of $K^{((0))}$ and a simplicial quotient map $f : K^{((0))} \to K'^{((0))}$ that abides by $\mathcal{V}$. The induced map of this quotient map also abides by the induced partition $\mathcal{P}(\mathcal{V})$, meaning

$$\left\{|f|^{-1}(x) : x \in |K'|\right\} = \mathcal{P}(\mathcal{V})$$

**Homeomorphic Identification Space:** The identification space of $|K|$ and $\mathcal{P}(\mathcal{V})$ is homeomorphic to $|K'|$.

## 3.4 Simplicial Surfaces

When is the underlying space of a simplicial complex a surface?

**Simplicial Surface:** Let $K$ be a simplicial complex in $\mathbb{R}^n$. Then $|K|$ is a surface if and only if $K$ is a 2-complex such that

i) each 1-simplex is the face of precisely two 2-simplices, and
ii) the underlying space of the link of every 0-simplex of $K$ is a 1-sphere (homeomorphic to $S^1$).

If $K$ satisfies this condition, then $K$ is called a simplicial surface.

**Discarding the First Condition:** The second condition in the statement above is actually enough by itself. So, if the underlying space of the link of every 0-simplex of $K$ is a 1-sphere (homeomorphic to $S^1$), then $K$ is a simplicial surface.

**Triangulation:** Let $Q \in \mathbb{R}^n$ be a topological surface. A simplicial complex $K$ is said to triangulate $Q$ if there is a homeomorphism $h : |K| \to Q$. In this case, we say that $Q$ is triangulated by $K$ and $K$ together with the homeomorphism $h$ is called a triangulation of $K$.

**Existence:** Any compact topological surface in $\mathbb{R}^n$ can be triangulated.

**Common Triangulations:** If a topological surface is triangulated by two simplicial surfaces $K_1$ and $K_2$, $K_1$ and $K_2$ have simplicially isomorphic subdivisions.

## 3.5 The Euler Characteristic

**Euler Characteristic of a Simplicial Complex:** Let $K$ be a 2-complex, and denote

$$V = \left|K^{((0))}\right| \qquad E = \left|K^{((1))}\right| \qquad F = \left|K^{((2))}\right|$$

Then the Euler characteristic of $K$, denoted $\chi(K)$ is defined as

$$\chi(K) = V - E + F$$

**Invariance under Triangulations:** Let $K_1$ and $K_2$ be two 2-complexes that triangulate the same compact surface $Q \subset \mathbb{R}^n$. Then

$$\chi(K_1) = \chi(K_2)$$

The proof of this statement uses the fact that if a complex $L$ is a subdivision of a complex $K$, then $\chi(K) = \chi(L)$, but the reason for this will be apparent later on.

**Euler Characteristic of a Surface:** The Euler characteristic of a compact topological surface is that of a 2-complex that triangulates it. This definition is well-defined due to the above fact.

**Connected Sum:** Let $Q_1$ and $Q_2$ be compact connected surfaces in $\mathbb{R}^n$. Then

$$\chi(Q_1 \# Q_2) = \chi(Q_1) + \chi(Q_2) - 2$$

## 3.6 Simplicial Curvature and the Simplicial Gauss-Bonnet Theorem

**Simplicial Curvature:** Let $K$ be a simplicial surface. The curvature of $K$ at a vertex (0-simplex) $v$ is defined as

$$d(v) = 2\pi - \sum_{\eta:\ v \in \eta} \angle(v, \eta)$$

The simplicial curvature $d(v)$, also called the angle defect of $v$, is a measure of how much the complex deviates from being flat at $v$.

**Simplicial Gauss-Bonnet Theorem:** Let $K$ be a simplicial surface. Then

$$\sum_{v \in K^{((0))}} d(v) = 2\pi\chi(K)$$

This simple statement is used in the proof of several important statements made earlier without creating any circular argumentation.

**Orientability and Euler Characteristic Characterization:** The Classification Theorem gives us an explicit list of all the compact connected surfaces, where the ones in the $T^2$ group are all orientable and those in the $P^2$ group are all non-orientable. If one examines the list, those in the same group all have different Euler characteristics. Therefore we conclude that two compact and connected topological surfaces $Q_1$ and $Q_2$ are homeomorphic if and only if they have the same Euler characteristic AND they are either both orientable or both non-orientable.

**Invariance of $\chi()$ under Subdivisions:** If the simplicial surface $L$ is (or is homeomorphic to) a subdivision of a simplicial surface $K$, then $\chi(K) = \chi(L)$.