MATH406 Cheat Sheet

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Last Updated: 27/06/2020

Contents

1 Languages and Structures

1.1 Languages

Language: A language $\mathcal{L}$ is a collection of symbols, which are separated into the following categories:

1. Logical symbols
2. Punctuation symbols
3. Variable symbols $Vars_{\mathcal{L}}$ (as many as you need)
4. Constant symbols $C_{\mathcal{L}}$ (optional)
5. Function symbols $F_{\mathcal{L}}$ (optional)
6. Relation symbols $R_{\mathcal{L}}$ (optional)

Because the first three are obligatory, you only give the symbols from the last three categories, if you have any, when specifying a language.

Term: For a given language $\mathcal{L}$, the terms of that language $Term_{\mathcal{L}}$ are defined recursively as follows:

TB1 Each variable is a term.

TB2 Each constant is a term.

TI If $t_1, t_2, ...t_n$ are terms and f is an n-ary function, then $f(t_1, t_2, ..., t_n)$ is also a term.

Formulas: For a given language $\mathcal{L}$, the formulas of that language $Form_{\mathcal{L}}$ are defined recursively as follows:

FB1 $t = u$ is a formula, where t and u are terms.

FB2 $r(t_1, t_2, ..., t_n)$ is a formula, where $t_1, t_2, ...t_n$ are terms.

FI1 If α is a formula, then so is $\neg\alpha$.

FI2 If α and β are formulas, then so is $\alpha \vee \beta$.

FI3 If x is a variable and φ is a formula, then so is $\forall x(\varphi)$.

Notice that for the inductive steps, the other logical connectives and $\exists$ are skipped. This is because they can be expressed in terms of the existing clauses.

Induction on the Complexity of Formulas: Much like induction on natural numbers,

- you prove it for a base case (atomic formulas),
- and show it for a non-atomic formula, assuming that the claim holds for the non-atomic formula's constituents.

Freedom: Given x a variable and φ a formula, we say that x is free in φ in the following cases:

- φ is atomic and x occurs in φ as a variable.
- φ is of the form $\neg\alpha$ and x occurs in α as a variable.
- φ is of the form $\alpha \vee \beta$ and x occurs in α or β as a variable.
- φ is of the form $\forall\Box\ \psi$, x is not $\Box$ and is free in ψ.

A formula with no free variables is called as a *sentence*.

Substitution: Given $u, t \in Term_{\mathcal{L}}$ and $x \in Vars_{\mathcal{L}}$, we define *the substitution of t for x in u*, u^x_t, inductively as follows:

- If u is a variable equal to x, then u^x_t is t.
- If u is a variable not equal to x, then u^x_t is u.
- If u is of the form $f(v_1, v_2, ..., v_n)$ with variables $v_1, v_2, ..., v_n$, then u^x_t is $f((v_1)^x_t, (v_2)^x_t, ..., (v_n)^x_t)$.

Given $\varphi \in Form_{\mathcal{L}}$, $t \in Term_{\mathcal{L}}$ and $x \in Vars_{\mathcal{L}}$, we define *the substitution of t for x in φ*, φ^x_t, inductively as follows:

- If φ is of the form $t_1 = t_2$, then φ^x_t is $(t_1)^x_t = (t_2)^x_t$.
- If φ is of the form $r(v_1, v_2, ..., v_n)$ with variables $v_1, v_2, ..., v_n$, then φ^x_t is $r((v_1)^x_t, (v_2)^x_t, ..., (v_n)^x_t)$.
- If φ is of the form $\neg\alpha$, then φ^x_t is $\neg\alpha^x_t$.
- If φ is of the form $\alpha \vee \beta$, then φ^x_t is $\alpha^x_t \vee \beta^x_t$.
- If φ is of the form $\forall\Box\ \psi$, then φ^x_t is φ if $\Box$ is x, and is $\forall\Box\ \psi^x_t$ otherwise.

Substitutability: Let $\varphi \in Form_{\mathcal{L}}$, $t \in Term_{\mathcal{L}}$ and $x \in Vars_{\mathcal{L}}$. We say that t *is substitutable for x in φ* if

- φ is atomic, or
- φ is of the form $\neg\alpha$ and is substitutable for x in α, or
- φ is of the form $\alpha \vee \beta$ and is substitutable for x in α and β, or
- φ is of the form $\forall y\ \psi$ and either
  - x is not free in φ, or
  - y does not occur in t and t is substitutable for x in α.

1.2 Structures

Fix a language $\mathcal{L}$.

Structure: An $\mathcal{L}$-structure $\mathfrak{A}$ consists of a

- Non empty set A which is the domain of the structure,
- An interpretation of the language $\mathcal{L}$, which means
  - for each constant symbol c of $\mathcal{L}$, an element $c^{\mathfrak{A}} \in A$,
  - for each n-ary function symbol f of $\mathcal{L}$, a function $f^{\mathfrak{A}} : A^n \to A$,
  - for each n-ary relation symbol r of $\mathcal{L}$, an element $r^{\mathfrak{A}} \subseteq A \times A \times ... \times A$.

The interpretation is what gives "meaning" to the language at hand.

Variable and Term Assignment Maps: Let $\mathfrak{A}$ be a structure. A variable assignment function is a mapping $s : Vars_{\mathfrak{A}} \to A$.
The modified variable assignment function $s[x|a]$ for some $a \in A$ and $x \in Vars_{\mathfrak{A}}$ is defined as

$$s[x|a](v) = \begin{cases} s(v) & ,v \neq x \\ a & ,v = a \end{cases}$$

The modified variable assignment function is useful in instantiating a variable to a specific value.
The term assignment function $\bar{s} : Term_{\mathcal{L}} \to A$ generated by s is defined recursively as follows:

- If t is a constant symbol, say c, then $\bar{s}(t) = c^{\mathfrak{A}}$,
- If t is a variable symbol, then $\bar{s}(t) = s(t)$,
- If t is of the form $f(t_1, t_2, ..., t_n)$ where $t_1, t_2, ..., t_n$ are terms, then $\bar{s}(t) = f^{\mathfrak{A}}(\bar{s}(t_1), \bar{s}(t_2), ..., \bar{s}(t_n))$.

Satisfaction (Tarski): Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, s be variable assignment map. We say that $\mathfrak{A}$ *satisfies* the formula φ with assignment s, written as $\mathfrak{A} \models \varphi[s]$, if

- φ is of the form $t = u$ and $\bar{s}(t) = \bar{s}(u)$, where t and u are terms.
- φ is of the form $r(t_1, t_2, ..., t_n)$ and $(\bar{s}(t_1), \bar{s}(t_2), ..., \bar{s}(t_n)) \in r^{\mathfrak{A}}$, where $t_1, t_2, ..., t_n$ are terms.
- φ is of the form $\neg\alpha$ and $\mathfrak{A} \not\models \alpha[s]$.
- φ is of the form $\alpha \vee \beta$ and $\mathfrak{A} \models \alpha[s]$ or $\mathfrak{A} \models \beta[s]$.
- φ is of the form $\forall x\alpha$ and for each element $a \in A$, we have $\mathfrak{A} \models \varphi[s[x|a]]$.

In case a formula φ is satisfied by a structure $\mathfrak{A}$ under any variable assignment map, then we write $\mathfrak{A} \models \varphi$.

Theory: A theory is a set of sentences $\Sigma \subseteq Form_{\mathcal{L}}$. In case a structure $\mathfrak{A}$ satisfies every sentence of the theory Σ, we write $\mathfrak{A} \models \Sigma$.

Theory of a Structure: Let $\mathfrak{A}$ be a structure. Its theory $Thm(\mathfrak{A})$ is defined as the set of sentences it satisfies.

$$Th(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi \text{ and } \varphi \in Form_{\mathcal{L}} \text{ is a sentence}\}$$

A similar definition holds for a set of sentences $\Sigma \subseteq Form_{\mathcal{L}}$.

Isomorphism of Structures: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures. We say that they are isomorphic, $\mathfrak{A} \cong \mathfrak{B}$, if there is a bijection $j : A \to B$ such that

- $j(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for every $c \in C_{\mathcal{L}}$.
- $(a, b, ..., c) \in r^{\mathfrak{A}}$ iff $(j(a), j(b), ..., j(c)) \in r^{\mathfrak{B}}$ for every $r \in R_{\mathcal{L}}$ and $a, b, ..., c \in A$.
- $f^{\mathfrak{A}}(a, b, ..., c) = f^{\mathfrak{B}}(j(a), j(b), ..., j(c))$ for every $f \in F_{\mathcal{L}}$ and $a, b, ..., c \in A$.

Equivalence of Structures: Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are said to be elementarily equivalent, $\mathfrak{A} \equiv \mathfrak{B}$ if they have the same theory: $Thm(\mathfrak{A}) = Thm(\mathfrak{B})$.
If two structures are isomorphic, they are elementarily equivalent. The converse need not be true.

Logical Consequences/Implications: Let $\Sigma, \Delta \in Form_{\mathcal{L}}$. We say that Σ *logically implies* Δ, $\Sigma \models \Delta$, if for any $\mathcal{L}$-structure $\mathfrak{A}$, if $\mathfrak{A} \models \Sigma$ then $\mathfrak{A} \models \Delta$. This means that any model $\mathfrak{A}$ of Σ is also a model of Δ.
A similar definition can be made between single formulas and sets of formulas as well: $\varphi \models \psi$, $\varphi \models \Sigma$, $\Sigma \models \varphi$.

2 Deductions

2.1 Intermission: Propositional Logic

Propositional Formulas: P, Q, R, ... are propositional variables and $\neg, \wedge, \vee, \to, \leftrightarrow$ are logical connectives. A propositional formula is defined inductively as follows:

- A propositional variable is a propositional formula (a.k.a. an atomic formula).
- If φ and ψ are propositional formulas, then so are the following:

$$\neg\varphi \qquad \varphi \vee \psi \qquad \varphi \wedge \psi \qquad \varphi \to \psi \qquad \varphi \leftrightarrow \psi$$

Given a formula $\varphi \in Form_{\mathcal{L}}$ in first order logic, one can obtain its propositional counterpart $\tilde{\varphi}$ easily.

Truth Assignment: A truth assignment map for propositional variables $v : PropVars \to \{0, 1\}$ can be easily extended to a map for propositional formulas $\bar{v} : PropForm \to \{0, 1\}$

as follows:

$$\bar{v}(\varphi) = \begin{cases} v(\varphi) & \text{if } \varphi \text{ is atomic} \\ 1 - v(\psi) & \text{if } \varphi \text{ is of the form } \neg\psi \\ v(\psi)v(\gamma) & \text{if } \varphi \text{ is of the form } \psi \wedge \gamma \\ \lceil (v(\psi) + v(\gamma))/3 \rceil & \text{if } \varphi \text{ is of the form } \psi \vee \gamma \\ \lceil ((1 - v(\psi)) + v(\gamma))/3 \rceil & \text{if } \varphi \text{ is of the form } \psi \rightarrow \gamma \\ 1 & \text{if } \varphi \text{ is of the form } \psi \leftrightarrow \gamma \text{ and } v(\psi) = v(\gamma) \end{cases}$$

Tautology: A propositional formula φ is a tautology if $\bar{v}(\varphi) = 1$ under any truth assignment map $\bar{v} : PropForm \rightarrow \{0, 1\}$.

Propositional Consequence/Implication: Given a set of propositional formulas $\tilde{\Gamma}$ and a propositional formula $\tilde{\varphi}$, we say that $\tilde{\varphi}$ is a propositional consequence of $\tilde{\Gamma}$ if for every $\bar{v} : PropForm \rightarrow \{0, 1\}$ with $\bar{v}(\tilde{\psi}) = 1$ for all $\tilde{\psi} \in \tilde{\Gamma}$, we have $\bar{v}(\tilde{\varphi}) = 1$. Equivalently, if the following propositional formula is a tautology:

$$\left(\bigwedge_{\tilde{\psi} \in \tilde{\Gamma}} \tilde{\psi} \right) \rightarrow \tilde{\varphi}$$

Given $\Gamma \subseteq Form_{\mathcal{L}}$ and $\varphi \in Form_{\mathcal{L}}$, one can easily extend the propositional implication definition here, by obtaining $\tilde{\Gamma}$ and $\tilde{\varphi}$.

2.2 Our Proof System

What does it mean to prove something? What is a proof?

Fix a language $\mathcal{L}$.

Proofs and Proof Systems: Let $\Sigma, \Lambda \subseteq Form_{\mathcal{L}}$. A finite sequence of $\mathcal{L}$-formulas $(\varphi_1, \varphi_2, ..., \varphi_n, \Phi)$ is said to be a deduction/proof of Φ, $\Sigma \vdash \Phi$, if for all φ_i we have

- $\varphi \in \Sigma$, or (Σ is the set of non-logical axioms)
- $\varphi \in \Lambda$, or (Λ is the set of logical axioms)
- There is a rule of inference (Γ, φ_i) such that $\Gamma \subseteq \{\varphi_1, \varphi_2, ..., \varphi_{n-1}\}$.

For a different selection of Σ, Λ and inference rules, we obtain a different *proof system*.

Our Logical Axioms: Our logical axioms Λ are of two types:

EA Equality Axioms: - For each $x \in Vars_{\mathcal{L}}$, $x = x$ is an axiom.

- For all variables x_1, x_2, ... x_n, y_1, y_2, ... y_n and an n-ary function symbol f, the following is an axiom:

$$[(x_1 = y_1) \wedge (x_2 = y_2) \wedge ... \wedge (x_n = y_n)] \rightarrow [f(x_1, x_2, ...x_n) = f(y_1, y_2, ...y_n)]$$

- For all variables x_1, x_2, ... x_n, y_1, y_2, ... y_n and an n-ary relation symbol r, the following is an axiom:

$$[(x_1 = y_1) \wedge (x_2 = y_2) \wedge ... \wedge (x_n = y_n)] \rightarrow [r(x_1, x_2, ...x_n) \leftrightarrow r(y_1, y_2, ...y_n)]$$

QA Quantifier Axioms: For each $t \in Term_{\mathcal{L}}$, $x \in Vars_{\mathcal{L}}$ and $\varphi \in Form_{\mathcal{L}}$, if t is substitutable for x in φ, then the following two are axioms:

UI Universal Instantiation: $\forall x\varphi \to \varphi^x_t$

EG Existential Generalization: $\varphi^x_t \to \exists x\varphi$

Our Inference Rules: There are two types of inference rules:

QR Quantifier Rules: For any two formulas φ and ψ and variable symbol x not free in ψ, we have

$(\{\psi \to \varphi\}, \psi \to \forall x\varphi)$

$(\{\varphi \to \psi\}, \exists x\varphi \to \psi)$

PC Propositional Consequences: For any finite set $\Gamma \subseteq Form_{\mathcal{L}}$ and $\varphi \in Form_{\mathcal{L}}$, if φ is a propositional consequence of Γ, then (Γ, φ) is a rule of inference.

Validity: A first-order formula $\varphi \in Form_{\mathcal{L}}$ is called *valid* if $\varnothing \models \varphi$, in which case we write $\models \varphi$. This means that it is true by itself, without needing any other formulas (axioms). Thus, valid statements are present in the theory of all structures.
If $\tilde{\varphi}$ is a tautology, then φ is valid. However, the converse is not true in general.

Two Technical Lemmas:

1. The logical axioms (EA and QA) of our proof system are valid.
2. Let (Γ, φ) be an inference rule. Then $\Gamma \models \varphi$.

Important Lemma: Let $C \subseteq Form_{\mathcal{L}}$. If C satisfies

- $\Sigma \subseteq C$,
- $\Lambda \subseteq C$, and
- If (Γ, φ) is an inference rule with $\Gamma \subseteq C$, then $\varphi \in C$,

then $Thm(\Sigma) \subseteq C$.

Soundness Theorem: Let $\Sigma \subseteq Form_{\mathcal{L}}$. For all $\varphi \in Form_{\mathcal{L}}$, if $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$.
This theorem is important, as it ties the syntactic notion of provability to the semantic notion of satisfaction.

Properties of Our Proof System:

Equality Properties: Let $x, y, z \in Vars_{\mathcal{L}}$. Then

$\vdash x = x$

$\vdash x = y \to y = x$

$\vdash (x = y \land y = z) \to x = z$

Universal Closure: $\Sigma \vdash \theta$ if and only if $\Sigma \vdash \forall x\theta$.

Deduction Theorem: Let $\Sigma \subseteq Form_{\mathcal{L}}$ and $\theta \in Sent_{\mathcal{L}}$. Then $\Sigma \cup \{\theta\} \vdash \varphi$ if and only if $\Sigma \vdash (\theta \to \varphi)$.

3 Completeness and Compactness

3.1 Completeness

Consistency and Inconsistency: Let $\Sigma \subseteq Form_{\mathcal{L}}$. Σ is said to be inconsistent if $\Sigma \vdash (\varphi \land \neg\varphi)$ for some φ. Σ is said to be consistent if it is not inconsistent, in which case we write

$Cons(\Sigma)$.
An inconsistent Σ can prove any $\varphi \in Form_{\mathcal{L}}$.
Σ is said to be maximally consistent if it contains all formulas φ which does not make it inconsistent, i.e. any addition to it from $Form_{\mathcal{L}} - \Sigma$ makes it inconsistent.

Gödel's Completeness Theorem: Let $\Sigma \subseteq Form_{\mathcal{L}}$. For all $\varphi \in Form_{\mathcal{L}}$, if $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

Second equivalent version: Let $\Sigma \subseteq Form_{\mathcal{L}}$. Then if Σ is consistent, it has a model.

This theorem is important, as it ties the semantic notion of satisfaction to the syntactic notion of provability. Combined with the Soundness Theorem, we obtain

$$\Sigma \models \varphi \text{ if and only if } \Sigma \vdash \varphi$$

3.2 Compactness

Satisfiability: $\Sigma \in Form_{\mathcal{L}}$ is said to be satisfiable if it has a model.

Finite Satisfiability: $\Sigma \in Form_{\mathcal{L}}$ is said to be finitely satisfiable if every finite subset of Σ has a model.

Compactness Theorem: Let $\Sigma \in Form_{\mathcal{L}}$. Σ is satisfiable if and only if it is finitely satisfiable.

4 Changing Models

4.1 (Elementary) Substructures, and the Tarski-Vaught Test

Substructure: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures. We say that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if

- $A \subseteq B$,
- For every constant symbol $c \in C_{\mathcal{L}}$, we have $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.
- For every n-ary relation symbol $r \in R_{\mathcal{L}}$, we have $r^{\mathfrak{A}} = r^{\mathfrak{B}} \cap A^n$, that is, for every $(a_1, a_2, \ldots, a_n) \in A^n$, $r^{\mathfrak{A}}(a_1, a_2, \ldots, a_n)$ if and only if $r^{\mathfrak{B}}(a_1, a_2, \ldots, a_n)$
- For every n-ary function symbol $f \in F_{\mathcal{L}}$, we have $a^{\mathfrak{A}} = f^{\mathfrak{B}} \restriction A^n$, that is, for every $(a_1, a_2, \ldots, a_n) \in A^n$, $f^{\mathfrak{A}}(a_1, a_2, \ldots, a_n) = f^{\mathfrak{B}}(a_1, a_2, \ldots, a_n)$

Let $\mathfrak{A}$ be a structure with domain A and $B \subseteq A$. B is the domain of a substructure if and only if $c^{\mathfrak{A}} \in B$ for all constant symbols and B is closed under $f^{\mathfrak{A}}$ for all function symbols in $\mathcal{L}$.
A substructure can behave very differently than its parent structure.

Elementary Substructure: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models such that $\mathfrak{A} \subseteq \mathfrak{B}$. We say that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$, written $\mathfrak{A} \preceq \mathfrak{B}$ if for every formula $\varphi(\bar{x})$ and $\bar{a} \in A$, we have $\mathfrak{A} \models \varphi(\bar{a})$ if and only if $\mathfrak{B} \models \varphi(\bar{a})$.

Embeddings: Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. A map $\pi : A \to B$ is an $\mathcal{L}$-embedding if

- π is one-to-one,
- For all n-ary relation symbols r and $\bar{a} \in A$, we have $r^{\mathfrak{A}}(\bar{a})$ if an only if $r^{\mathfrak{B}}(\pi(\bar{a}))$,
- For all n-ary function symbols f and $\bar{a} \in A$, we have $\pi(f^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(\pi(\bar{a}))$.

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$-isomorphism.

Elementary embedding: Let $\pi : A \to B$ be an L-embedding. We say that π is an elementary embedding if the substructure $\mathfrak{B}$ with domain $\pi(A)$ is an elementary substructure.

Tarski-Vaught Test: LEt $\mathfrak{A} \subseteq \mathfrak{B}$ be two $\mathcal{L}$-structures. Suppose for every formula $\varphi(\bar{x}, y)$ and for every $\bar{a} \in A$ and $b \in B$, if $\mathfrak{B} \models \varphi(\bar{a}, b)$ then there is some $d \in A$ with $\mathfrak{B} \models \varphi(\bar{a}, d)$. Then $\mathfrak{A} \preceq \mathfrak{B}$.
It is very difficult to make use of the Tarski-Vaught test on its own. For us, it is mostly important in the proofs of the Löwenheim-Skolem theorems.

4.2 Löwenheim-Skolem Theorems

Downward Löwenheim-Skolem Theorem: Let $\mathfrak{B}$ be an $\mathcal{L}$-structure and let $S \subseteq B$. Then there is an elementary substructure $\mathfrak{A} \preceq \mathfrak{B}$ such that $S \subseteq A$ and $|A| = \max\{|S|, |\mathcal{L}|, \aleph_0\}$.
Setting $S = \varnothing$, if your language is countable, this theorem guarantees the existence of countable elementary substructures.

Complete Diagrams: Ler $\mathfrak{A}$ be an $\mathcal{L}$-structure. Expand $\mathcal{L}$ into $\mathcal{L}_{\mathfrak{A}}^{+} = \mathcal{L} \cup \{c_a : a \in A\}$, where c_a's are new distinct constant symbols. Expand $\mathfrak{A}$ into $\mathfrak{A}^{+}$ by interpreting $c_a^{\mathfrak{A}} = a$ for all $a \in A$. The complete diagram of $\mathfrak{A}$ is defined as $Th(\mathfrak{A}^{+})$, denoted as $CDiag(\mathfrak{A})$.
The complete diagram of a structure contains any and all sentences within that structure regarding every single element, and hence is a wider structure.

Important Lemma: Let $\mathfrak{A}$, $\mathfrak{B}$ be two $\mathcal{L}$-structures. If $\mathfrak{B}$ can be expanded into a $\mathcal{L}_{\mathfrak{A}}^{+}$ structure $\mathfrak{B}^{+}$ such that $\mathfrak{B}^{+} \models CDiag(\mathfrak{A})$, then there is an elementary embedding $\pi : A \to B$.

Upward Löwenheim-Skolem Theorem: For any infinite structure $\mathfrak{A}$ and a cardinal $\kappa \geq \max\{|A|, |\mathcal{L}|, \aleph_0\}$, there is an $\mathcal{L}$-structure $\mathfrak{B}$ such that $\mathfrak{A} \preceq \mathfrak{B}$ and $|B| = \kappa$.
Corollary: Let Σ be a set of sentences over a countable language. If Σ has infinite models, then Σ has models of *all* infinite cardinals.

Skolem's Paradox: Suppose that ZFC is consistent. Then by completeness, it must have a model, $\mathcal{M} = (M, \in^{\mathcal{M}}) \models ZFC$. Let us suppose further that M is transitive and $\in^{\mathcal{M}} = \{(a, b) : a \in b; a, b \in M\}$. As $\mathcal{L} = \{\in\}$ is countable, by the downward Löwenheim-Skolem theorem, there is some $\mathfrak{A} \preceq (M, \in^{\mathcal{M}})$ with $|A| = \aleph_0$. But since $(A, \in) \models ZFC$, we must have $(A, \in) \models \neg\exists f$ 'f is a bijection from $\mathbb{N}$ to $\mathbb{R}'$, but A is countable. So, how does that happen?
In reality, there are such bijections, but none of them are in $\mathfrak{A}$. For this reason $\mathfrak{A}$ 'believes' that there are no bijections from $\mathbb{N}$ to $\mathbb{R}$, whereas, in reality, both $\mathbb{N}^{\mathfrak{A}}$ and $\mathbb{R}^{\mathfrak{A}}$ are countable.

4.3 Complete Theories, Categoricity, and Back-and-Forth

Completeness of a Theory: A theory Σ is said to be complete if for every $\mathcal{L}$-sentence φ, either $\Sigma \vdash \varphi$ ($\Sigma \models \varphi$) or $\Sigma \vdash \neg\varphi$ ($\Sigma \models \neg\varphi$).

Categoricity: Let κ be a cardinal number. A theory Σ is said to be κ-categorical if

- Σ has models of cardinality κ, and
- Any two models of Σ of cardinality κ are isomorphic.

Łoś–Vaught Test: Let Σ be κ-categorical for some $\kappa \geq \max\{\aleph_0, |\mathcal{L}|\}$ and suppose that Σ does not have finite models. Then Σ is complete.

Back-and-forth Method: The back-and-forth method is a method of proof. Here, it was used to prove that DLOWE is $\aleph_0$-categorical. In the proof, to show categoricity, one must construct an isomorphism between two random DLOWE structures. One starts with two singletons from each of the two models, and goes 'back and forth' between these two by adding elements to both of them while still conserving the isomorphism.

DLOWE: Let $\mathcal{L} = \{<\}$. The theory of Dense Linear Orders Without Endpoints (DLOWE) is a theory of this language as given below.

$$\begin{aligned}
&\forall x\ \neg(x < x) && \text{(Strict Linear Order: Irreflexive)}\\
&\forall x \forall y \forall z\ ((x < y \wedge y < z) \to x < z) && \text{(Strict Linear Order: Transitive)}\\
&\forall x \forall y\ ((x < y) \vee (x = y) \vee (y < x)) && \text{(Strict Linear Order: Trichotomy)}\\
&\forall x \forall y (x < y \to \exists z\ (x < z \wedge z < y)) && \text{(Denseness)}\\
&\neg \exists x \forall y (x = y \vee y < x) && \text{(No Greatest Element)}\\
&\neg \exists x \forall y (y = x \vee x < y) && \text{(No Least Element)}
\end{aligned}$$

DLOWE clearly has no finite models. Being $\aleph_0$-categorical, by Łoś–Vaught test, it must be complete.

4.4 Definability

Definability: Let $\mathfrak{A}$ be an $\mathcal{L}$-structure. A subset $X \subseteq A^n$ is called S-definable if there exists an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in S \subseteq A$ such that $X = \{\bar{a} : \mathfrak{A} \models \varphi(\bar{a}, \bar{b})\}$.
Here the set S is a set of parameters to define X. If we let $S = \varnothing$, we call such sets X as definable without parameters or just definable.

Lemma (Definable projections): If $X \subseteq A^{n+m}$ is S-definable, then so is the projection $proj(X) = \{\bar{a} \in A^n : \text{There is some } \bar{b} \in A^m \text{ such that } (\bar{a}, \bar{b}) \in X\}$.

Theorem (Pointwise fixing): If $X \subseteq A^n$ is S-definable and $f : A \to A$ is an automorphism such that f fixes S pointwise (i.e. $f(s) = s$ for all $s \in S$), then f fixes X setwise ($X = \{f(x) : x \in X\} = f[X]$).
The converse of this theorem is also quite useful when $S = \varnothing$. Then the pointwise fixing condition is vacuously satisfies, and the theorem is only a relation between definability and the setwise fixing ability of an automorphism. The converse is the following: If an automorphism of A does not fix a set X setwise, then X cannot be definable (without parameters.)

4.5 Quantifier Elimination and Model-Completeness

Quantifier Elimination for Structures: A structure $\mathfrak{A}$ is said to admit quantifier elimination if and only if for every $n \in \mathbb{N}^+$, every definable subset of A^n is defined by a quantifier-free formula.
We need such a concept because for some models, definable subsets can get very complicated (e.g. $(\mathbb{N}, +, \cdot, 0, 1)$). Understanding definable subsets defined by quantifier-free formulas is generally an easier task.

Principle DLO Formulas: A principle DLO formula for the variables $x_1, x_2, ..., x_n$ is a formula $\psi(x_1, x_2, ..., x_n)$ of the form

- If $n = 1$, $\psi(x_1)$ is $x_1 = x_1$.

- If $n > 1$, $\psi(x_1, x_2, ..., x_n)$ is $\bigwedge_{i=1}^{n-1} x_{\sigma(i)} \; \Box_i \; x_{\sigma(i+1)}$ where $\sigma \in Sym\{1, 2, ..., n\}$ is a permutation and $\Box_i$ is either $<$ or $=$.

A principle DLO formula basically tells the configuration of a certain number of elements. The job of σ is to tell which two variables are related consecutively and the job of $\Box_i$ at each i-th step is to determine the relation between the two elements.
A Lemma: Let $\psi(\bar{x})$ be a principle DLO formula and $\bar{a}, \bar{b} \in \mathbb{Q}^n$ such that $(\mathbb{Q}, <) \models \psi(\bar{a}) \wedge \psi(\bar{b})$. Then there exists $f \in Aut(\mathbb{Q}, <)$ with $f(\bar{a}) = \bar{b}$.
This means that for the same specific configuration of certain elements (as set by ψ), one can transform two groups of numbers satisfying that condition to the other via some automorphism f. Think of f as a collection of rules that stretches and translates the rational line to 'tailor' one group of number to the other.
Important: $(\mathbb{Q}, <)$ admits quantifier elimination.

Quantifier Elimination for Theories: A theory T is said to admit quantifier elimination if for every formula φ, there exists a quantifier-free formula ψ such that $T \vdash \varphi \leftrightarrow \psi$ ($T \models \varphi \leftrightarrow \psi$).
Important: DLO admits quantifier elimination.

Model-Completeness: A theory T is called model-complete if $\mathfrak{A} \preceq \mathfrak{B}$ whenever $\mathfrak{A}, \mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}$. That is to say, any substructure of $\mathfrak{B} \models T$ is an elementary substructure.

Quantifier Elimination & Model-Completeness: If a theory T admits quantifier elimination, then T is model-complete.

4.6 Ultraproducts, Łoś' Theorem and Some Applications

Filter: Let X be a set. A filter $\mathcal{F} \subseteq \mathcal{P}(X)$ on X is a collection such that

- $\varnothing \notin \mathcal{F}$ and $X \in \mathcal{F}$
- For all $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$
- For all $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, $B \in \mathcal{F}$

A filter sort of 'chooses' or 'prefers' certain subsets of X. But we can think of a filter that does not include both some subset Y and its complement Y^C. Then is Y 'preferred' by this filter or not?

Ultrafilter: A filter $\mathcal{U}$ on a set X is an ultrafilter if for all $A \subseteq X$, we have either $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$.
Now an ultrafilter becomes a filter that decisively divides the $\mathcal{P}(X)$ into two parts, and we can judge *all* subsets of X on whether they are preferred or not (not being preferred meaning that its complement is preferred).

Principle Ultrafilter: $\mathcal{U} = \{Y \subseteq X | a \in Y\}$ is called as the principle ultrafilter generated by the element $a \in X$.

Filters to Ultrafilters: For every filter $\mathcal{F} \subseteq \mathcal{P}(X)$, there is an ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subseteq \mathcal{U}$.

Ultraproduct: Let $\mathcal{L}$ be a language with constant symbols $C_{\mathcal{L}}$, relation symbols $R_{\mathcal{L}}$ and function symbols $F_{\mathcal{L}}$. Let I be an index set and $\mathcal{U} \subseteq \mathcal{P}(I)$ an ultrafilter over the index set. Let $\{\mathfrak{M}_i | i \in I\}$ be a set of $\mathcal{L}$-structures indexed by I.
Define an equivalence relation $\sim$ on $\prod_{i \in I} \mathfrak{M}_i$ given by

$$(a_i)_{i \in I} \sim (b_i)_{i \in I} \text{ if and only if } \{i \in I | a_i = b_i\} \in \mathcal{U}$$

That is to say that the sequences $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are said to be related if and only if they agree on a "large portion" of their indices where the bigness is defined using the ultrafilter. Or in other words, they are equal if and only if they agree on the "preferable" indices. Using this equivalence relation, define the set M as

$$M = \prod_{i\in I} M_i/\sim$$

We will now create an $\mathcal{L}$-structure $\mathfrak{M}$ with domain M.
For each constant symbol $c \in C_{\mathcal{L}}$, define $r^{\mathfrak{M}}$ by

$$c^{\mathfrak{M}} = \overline{(c^{\mathfrak{M}_i})_{i\in I}}$$

where the overline denotes the congruence class. The notation means that each constant symbol is interpreted as the congruence class of the sequence constructed by the interpretation of that constant symbol in the set $\{\mathfrak{M}_i | i \in I\}$.
For each n-ary relation symbol $r \in R_{\mathcal{L}}$, define $r^{\mathfrak{M}}$ by

$$\left(\overline{(x_i)^1_{i\in I}}, \overline{(x_i)^2_{i\in I}}, ..., \overline{(x_i)^n_{i\in I}}\right) \in r^{\mathfrak{M}} \text{ if and only if } \{i \in I | (x_i^1, x_i^2, ..., x_i^n) \in r^{\mathfrak{M}_i}\} \in \mathcal{U}$$

This means that a group of sequences are related under r if and only if the indices at which the tuples formed by the elements of each sequence with that index, is "preferred" according to $\mathcal{U}$.
For each n-ary function symbol $f \in F_{\mathcal{L}}$, define $f^{\mathfrak{M}}$ by

$$f^{\mathfrak{M}}\left(\overline{(x_i)^1_{i\in I}}, \overline{(x_i)^2_{i\in I}}, ..., \overline{(x_i)^n_{i\in I}}\right) = \overline{\left(f^{\mathfrak{M}_i}(x_i^1, x_i^2, ..., x_i^n)\right)}$$

That is to say that the image of a group of sequences under f is the series congruent to the series made by the image of the function in each structure. This function is proved to be well-defined.
Then finally, the $\mathcal{L}$-structure

$$\mathfrak{M} = \left(\prod_{i\in I} \mathfrak{M}_i/\sim, \{f^{\mathfrak{M}}\}_{f\in F_{\mathcal{L}}}, \{r^{\mathfrak{M}}\}_{r\in R_{\mathcal{L}}}, \{c^{\mathfrak{M}}\}_{c\in C_{\mathcal{L}}}\right)$$

is called as the ultraproduct of $\{\mathfrak{M}_i | i \in I\}$ with respect to $\mathcal{U}$ and is denoted by

$$\mathfrak{M} = \prod_{i\in I} \mathfrak{M}_i/\mathcal{U}$$

Łoś' Theorem: Let $\varphi(x_1, x_2, ..., x_n)$ be an $\mathcal{L}$-formula and $\overline{(a_i)^1_{i\in I}}, \overline{(a_i)^2_{i\in I}}, ..., \overline{(a_i)^n_{i\in I}} \in \prod_{i\in I} M_i/\sim$.
Then

$$\prod_{i\in I} \mathfrak{M}_i/\mathcal{U} \models \varphi\left(\overline{(a_i)^1_{i\in I}}, \overline{(a_i)^2_{i\in I}}, ..., \overline{(a_i)^n_{i\in I}}\right)$$

if and only if

$$\{i \in I | \mathfrak{M}_i \models phi(a_i^1, a_i^2, ..., a_i^n)\} \in \mathcal{U}$$

Ultrapower: Let I be an index set and $\mathcal{U} \in \mathcal{P}(I)$ be an ultrafilter. Set $\mathfrak{M}_i = \mathfrak{M}$ for all $i \in I$.
The the ultraproduct $\prod_{i\in I} \mathfrak{M}_i/\mathcal{U} = \prod_{i\in I} \mathfrak{M}/\mathcal{U}$ is called as the ultrapower of $\mathfrak{M}$ with respect to $\mathcal{U}$.
The map $f : M \prod_{i\in I} M/\sim$ given by $f(m) = \overline{(m)_{i\in I}}$ is an elementary embedding from $\mathfrak{M}$ to $\prod_{i\in I} \mathfrak{M}/\mathcal{U}$