MATH 320 SET THEORY

MIDTERM #1

AXIOMS of ZFC

Empty Set: There exists a set to which no other set belongs.

$$\exists x \forall y \; y \notin x$$

Extensionality: Two sets are equal if and only if they have the same elements.

$$\forall x \forall y \,(x = y \leftrightarrow \forall z\, (z \in x \leftrightarrow z \in y))$$

Pairing: For any sets x & y, there exists a set z which consists of the elements x & y.

$$\forall x \forall y \,\exists z \,(t \in z \leftrightarrow t = x \lor t = y)$$

Union: For any set x, there exists a set y which consists of exactly the elements of elements of x.

$$\forall x \,\exists y \,\forall z \,(z \in y \leftrightarrow \exists s\,(s \in x \land z \in s))$$

Separation: Let $\varphi(z,p)$ be a formula in the language of set theory, with two variables z & p. For any p and for any x, there exists a set y that consists of elements of x satisfying $\varphi(\cdot, p)$.

$$\forall p \forall x \,\exists y \,\forall z \,(z \in y \leftrightarrow (z \in x \land \varphi(z,p)))$$

Power Set: For any set x, there exists a set y which consists of all subsets of x.

$$\forall x \,\exists y \,\forall z \,(z \in y \leftrightarrow z \subseteq x)$$

Choice: For all sets I and for indexed systems of sets $\{A_i\}_{i \in I}$ with $A_i \neq \phi$ for all $i \in I$, the product $\prod_{i \in I} A_i$ is non-empty.

Foundation: If S is a non-empty set, then there exists $s \in S'$ such that $s \cap S = \phi$

$$\forall S\,(S' \neq \phi \rightarrow (\exists s \in S' \; s \cap S' = \phi))$$

Infinity: An inductive set exists.

$$\exists x\,(\phi \in x \land (\forall y\,(y \in x \rightarrow S(y) \in x)))$$

Replacement: Let $\varphi(x,y)$ be a formula in the language of set theory with two free variables. If for all x there exists a unique y such that $\varphi(x,y)$, then for any A, there exists B such that for all y, we have $y \in B$ iff there exists $x \in A$ such that $\varphi(x,y)$.

$$(\forall x \exists y \forall y'\,(\varphi(x,y') \rightarrow y = y') \rightarrow (\forall A \,\exists B \,\forall y\,(y \in B \leftrightarrow \exists x\,(x \in A \land \varphi(x,y))))$$

IMPORTANT STUFF

Product of an indexed system: Let $\{F_i\}_{i\in I}$ be an indexed system of sets with the index set I. Then the product of the indexed system $\{F_i\}_{i\in I}$ is

$$\{f : I \to \bigcup\{F_i\}_{i\in I} \mid \forall i \in I,\ f(i) \in F_i\}$$

Equivalence Relation: For $E \in X \times X$, E is an equivalence relation if E is

reflexive:	$\forall a \in X,\ aEa$
symmetric:	$\forall a,b \in X,\ aEb \leftrightarrow bEa$
transitive:	$\forall a,b,c \in X,\ (aEb \wedge bEc) \to aEc$

All-or-nothing partitioning for equivalence relations: For $X \neq \emptyset$ and E an equivalence relation on X, then $\forall x,y \in X$, we have either

$$[x]_E = [y]_E \quad \text{or} \quad [x]_E \cap [y]_E = \emptyset$$

Transversals: Let E be an equivalence relation on X. A subset T is said to be a transversal for E if for every $x \in X$, there exists $y \in T$ such that $[x]_E \cap T = \{y\}$

Partial Orders: For $E \in X \times X$, E is a partial order if E is

reflexive:	$\forall a \in X,\ aEa$
transitive:	$\forall a,b,c \in X,\ (aEb \wedge bEc) \to aEc$
antisymmetric:	$\forall a,b \in X,\ (aEb \wedge bEa) \to a=b$

Strict Orders: For $E \in X \times X$, E is a strict order if E is

transitive:	$\forall a,b,c \in X\ (aEb \wedge bEc) \to aEc$
asymmetric:	$\forall a,b \in X\ aEb \to b\not{E}a$

Order extrema: For a partial order $\leq$ on X, an element x is a

- least element if $\forall y \in X,\ x \leq y$.
- minimal element if $\forall y \in X\ (y \leq x \to x = y)$.
- greatest element if $\forall y \in X\ y \leq x$.
- maximal element if $\forall y \in X,\ (x \leq y \to x = y)$

Linear order: If for a partial order $\leq$ on X, any two elements $a,b \in X$ are comparable, then $\leq$ is a linear order.

Well-order: If a partial order $\leq$ on some X is a linear order where every non-empty subset of X has a least element, $\leq$ is a well-order.

Predecessor: For $(S, \leq)$ and some $s \in S$, $\text{pred}(s) = \{x \in S : x < s\}$

Successor: For a well-order $(S, \leq)$ the least element of $\{t \in S : t > s\}$ is called the successor of s, denoted by s^+.

Initial segment: For linearly ordered $(S, \leq)$ and for $I \subseteq S$, I is called an initial segment of S if $\text{pred}(i) \subseteq I$ for every $i \in I$.

Order Isomorphism: $(P, \leq) \cong (Q, \preceq)$ if there exists a bijection $f: P \to Q$ such that for $p_1, p_2 \in P$, $p_1 < p_2 \to f(p_1) < f(p_2)$

Isomorphism Theorem for Well-Orders: Let $(P, \leq)$ & $(Q, \preceq)$ be well-ordered sets. Then either one of the three cases is true:

- $(P, \leq) \cong (Q, \preceq)$
- $(P, \leq)$ is isomorphic to some proper initial segment of Q.
- $(Q, \preceq)$ is isomorphic to some proper initial segment of P.

Well-founded relations: Let $E \subseteq X \times X$. E is well-founded if

$$\forall M \subseteq X \, (M \neq \emptyset \to (\exists m \in M \; \forall s \in M \; (s, m) \notin E))$$

which is to say that every non-empty subset of M has an E-minimal element.

Inductive set: A set X is inductive if $\emptyset \in X$ and the successor of any member of X is also a member of X.

Principle of Induction: Let $\varphi(n)$ be a property of sets. If $I = \{n \in \mathbb{N} : \varphi(n)\}$ is inductive, then $\mathbb{N} = I$.

Natural Numbers: By the axiom of infinity,

$$\mathbb{N} = \{x \in I : \forall J \, (\text{"}J \text{ is inductive"} \to x \in J)\}$$

Recursion Theorem: For $X \neq \emptyset$, $x \in X$ and $f: X \to X$, there exists a function $g: \mathbb{N} \to X$ such that

- $g(0) = x$
- $g(s(n)) = f(g(n)) \quad \forall n \in \mathbb{N}$

Finite Set: A set X is said to be finite if there exists a bijection $f: X \to n$ for some $n \in \mathbb{N}$.

Pigeonhole Principle: Let $m < n$, $m, n \in \mathbb{N}$. There is no injective function from n to m. Conversely, there is no surjective function from m to n.

Dedekind Infiniteness: For a Dedekind-infinite set X, there exists an injection from X to a proper subset of X.

Cardinality: For sets A & B,

$|A| \leq |B|$ if there exists an injection from A to B.

$|A| < B$ if there exists an injection but no bijection from A to B.

Cantor's Theorem: For any set X, $|X| < |\mathcal{P}(X)|$

Cantor-Schröder-Bernstein Theorem: For sets A & B, if there exists injections from A to B and from B to A, then there exists a bijection between A and B.

Integers: Define $\sim$ on $\mathbb{N} \times \mathbb{N}$ by

$$(p,q) \sim (r,s) \leftrightarrow p +_{\mathbb{N}} s = q +_{\mathbb{N}} r$$

Then

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$$

Rational numbers: Define $\sim$ on $\mathbb{Z} \times \mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, by

$$(p,q) \sim (r,s) \leftrightarrow p \cdot_{\mathbb{Z}} s = q \cdot_{\mathbb{Z}} r$$

Then

$$\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^* / \sim$$

Dedekind cut: $S \subseteq \mathbb{Q}$ is a Dedekind cut if

- $S \neq \emptyset$ & $S \neq \mathbb{Q}$
- S has no greatest element
- S is closed downwards.

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MIDTERM # 2

ORDINAL NUMBERS

Transitivity of a set: A set x is said to be transitive if every element of x is also a subset of x, i.e.

$$\forall y\,(y \in x \to y \subseteq x)$$

Ordinal number: A set α is is an ordinal number if

α is transitive
α is a strictly well-ordered set, ordered by $\in_\alpha$

For ordinals α, β, γ we have

i. If $\alpha \in \beta$ and $\beta \in \gamma$, then $\alpha \in \gamma$.
ii. If $\alpha \in \beta$, then $\beta \notin \alpha$.
iii. Either $\alpha \in \beta$, $\alpha = \beta$ or $\beta \in \alpha$.
iv. Every non-empty set of ordinals has a least element with respect to $\in$.

Successor and Limit ordinals: An ordinal α is said to be a successor ordinal if $\alpha = S(\beta)$ for some ordinal β. An ordinal α is said to be a limit ordinal if $\alpha \neq 0$ and α is not a successor ordinal.

Axiom of Replacement: Let $\varphi(x,y)$ be a formula in the language of set theory with two free variables. If for all x there exists a unique y such that $\varphi(x,y)$ holds, then for any A, there exists B such that for all y, we have $y \in B$ if and only if there exists $x \in A$ such that $\varphi(x,y)$ holds.

Class Function: Let $\varphi(x,y)$ be a formula in the language of set theory such that for all x, there exists a unique y such that $\varphi(x,y)$ holds. For each set x, we write

$$y = F_\varphi(x) \text{ if } \varphi(x,y) \text{ holds.}$$

and for a set A

$$F_\varphi[A] = \{y : \exists x \in A \ F_\varphi(x) = y\}$$

$$F\restriction_A = \{(x, F_\varphi(x)) : x \in A\}$$

Ordinals and strict orders: Let $(W, \prec)$ be a strictly well-ordered set. Then there exists a unique ordinal α such that $(W, \prec)$ and $(\alpha, \in_\alpha)$ are isomorphic.
The order type of $(W, \prec)$ is this unique ordinal α, denoted by

$$\mathrm{ot}(W, \prec) = \alpha$$

Hartogs numbers: By Thm. 31, there exists an ordinal λ such that there is no injection from λ to a set X.
For any given set X, the Hartogs number is defined to be the least of such ordinals, denoted by $\aleph(X)$. By definition;

$$\aleph(\omega) = \omega_1$$

which is the first uncountable ordinal.

Transfinite induction: Let $\varphi(x)$ be a formula in the language of set theory with one free variable. Assume that for all ordinals γ, we have that if $\varphi(\beta)$ for $\beta < \gamma$, then $\varphi(\gamma)$. Then $\varphi(\gamma)$ holds for all ordinals.

Transfinite induction alternative formulation. Let $\varphi(x)$ be a formula in the language of set theory with one free variable. If

i. $\varphi(0)$ holds
ii. If $\varphi(\gamma)$ holds for ordinal γ, it holds for $S(\gamma)$ so $\varphi(S(\gamma))$.
iii. For a limit ordinal θ, if $\varphi(\gamma)$ holds for all $\gamma < \theta$, then $\varphi(\theta)$ holds.

Then $\varphi(\cdot)$ holds for all ordinals.

Transfinite recursion: Let F_φ be a class function. Then there exists a class function F_ψ such that $F_\psi(\alpha) = F_\varphi(F_\psi \restriction_\alpha)$ for all ordinals α. This class function is unique in the sense that if there exists another class function F_ϕ satisfying the same property, then $F_\phi(\alpha) = F_\psi(\alpha)$ for all ordinals α.

Transfinite recursion alternative formulation: Let $F_{\varphi_1}, F_{\varphi_2}, F_{\varphi_3}$ be class functions. Then there exists a unique class function F_ψ such that

i. $F_\psi(0) = F_{\varphi_1}(0)$,
ii. $F_\psi(S(\alpha)) = F_{\varphi_2}(F_\psi(\alpha))$ for all ordinals α,
iii. $F_\psi(\alpha) = F_{\varphi_3}(F_\psi \restriction_\alpha)$ for all limit ordinals α.

Ordinal arithmetic: For ordinals β and γ

Addition:
$\beta + 0 = \beta$
$\beta + S(\gamma) = S(\beta + \gamma)$ for all ordinals γ
$\beta + \gamma = \sup\{\beta + \theta : \theta < \gamma\}$ for limit ordinal γ

Multiplication:
$\beta \cdot 0 = 0$
$\beta \cdot S(\gamma) = \beta \cdot \gamma + \beta$ for all ordinals γ
$\beta \cdot \gamma = \sup\{\beta \cdot \theta : \theta < \gamma\}$ for limit ordinals γ

Exponentiation:
$\beta^0 = 1$
$\beta^{S(\gamma)} = \beta^\gamma \cdot \beta$ for all ordinals γ
$\beta^\gamma = \sup\{\beta^\theta : \theta < \gamma\}$ for all limit ordinals γ.

Properties:

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$$
$$\alpha<\beta \longrightarrow \gamma+\alpha<\gamma+\beta$$
$$\alpha\leq\beta \longrightarrow \alpha+\gamma\leq\beta+\gamma$$
$$\gamma+\alpha=\gamma+\beta \longrightarrow \alpha=\beta$$
$$(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$$
$$\alpha.(\beta+\gamma) = \alpha.\beta+\alpha.\gamma$$
$$\alpha<\beta \longrightarrow \alpha.\gamma\leq\beta.\gamma$$
$$\text{If } \gamma>0, \text{ then } \alpha<\beta \longrightarrow \gamma.\alpha<\gamma.\beta$$
$$\alpha^{\beta+\gamma} = \alpha^{\beta}.\alpha^{\gamma}$$
$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta.\gamma}$$
$$\text{If } \gamma>1, \text{ then } \alpha<\beta \longrightarrow \gamma^{\alpha}<\gamma^{\beta}$$
$$\alpha<\beta \longrightarrow \alpha^{\gamma}\leq\beta^{\gamma}$$

Useful supremum property: Let X be a non-empty set of ordinals and α be an ordinal. Then

$$\sup\{\alpha+\beta : \beta\in X\} = \alpha+\sup\{\beta:\beta\in X\} = \alpha+\sup(X)$$
$$\sup\{\alpha.\beta : \beta\in X\} = \alpha.\sup(X)$$
$$\sup\{\alpha^{\beta} : \beta\in X\} = \alpha^{\sup(X)}$$

Intuition about addition and multiplication: For ordinals α and β

$\alpha+\beta$: "Attach β to the end of α"

$\alpha.\beta$: "Put β copies of α back to back"

Cantor normal form: Let $\alpha>0$ be an ordinal. Then there exists unique ordinals $\beta_1>\beta_2>\dots>\beta_n$ and positive natural numbers $k_1, k_2, \dots k_n$ such that

$$\alpha = \omega^{\beta_1}k_1 + \omega^{\beta_2}k_2 + \dots + \omega^{\beta_n}k_n$$

Lemmas for Cantor normal form:

Let α,γ be ordinals such that $\alpha\leq\gamma$. Then there exists a unique ordinal β such that $\gamma=\alpha+\beta$.

Let α,γ be ordinals such that $1\leq\alpha\leq\gamma$. Then there exists a greatest ordinal β such that $\alpha.\beta\leq\gamma$.

Let α be ordinals such that $2\leq\alpha\leq\gamma$. Then there exists a greatest ordinal β such that $\alpha^{\beta}\leq\gamma$.

Euclidean division for ordinals: Let α,γ be ordinals with $\gamma\neq 0$. Then there exists unique ordinals β and ρ with $\rho<\gamma$ such that

$$\alpha = \gamma.\beta+\rho$$

Let α,β be ordinals such that $\alpha<\beta$ and $k\in\omega$. Then $\omega^{\alpha}.k<\omega^{\beta}$

"Rules" for arithmetic with Cantor normal form:

1) For all ordinals α, γ; if $\alpha < \gamma$ then $\omega^{\alpha} + \omega^{\gamma} = \omega^{\gamma}$

2) Let $\alpha < \gamma$ be ordinals and $m, n \in \omega$ such that $n > 0$. Then $\omega^{\alpha} . m + \omega^{\gamma} . n = \omega^{\gamma} . n$

3) Let $\omega^{\beta_1} k_1 + \omega^{\beta_2} k_2 + \dots + \omega^{\beta_n} k_n$ be the Cantor normal form of a non-zero ordinal α. Then, for any $k \in \omega$ with $k > 0$, we have

$$\alpha . k = \omega^{\beta_1} (k_1 . k) + \omega^{\beta_2} k_2 + \dots + \omega^{\beta_n} k_n$$

4) Let $\omega^{\beta_1} k_1 + \omega^{\beta_2} k_2 + \dots + \omega^{\beta_n} k_n$ be the Cantor normal form of a non-zero ordinal α. Then for any ordinal $\gamma > 0$, we have

$$\alpha . \omega^{\gamma} = \omega^{\beta_1 + \gamma} \quad \text{(drop the other terms \underline{and} } k\text{s)}$$

MATH 320 SET THEORY

FINAL

CARDINAL NUMBERS

Zorn's Lemma: Let $(P, \leqslant)$ be a partially ordered set such that every linearly ordered subset (chain) has an upper bound in P; that is for every $C \subseteq P$, if P is a chain then there exists $p \in P$ such that $c \leqslant p$ for all $c \in C$. Then there exists a maximal element in P with respect to $\leqslant$. Then there exists a maximal element in P with respect to $\leqslant$.

Well-ordering theorem: Every set can be well-ordered, i.e. for every set A there exists a well order relation $\leqslant_A$ on A.

Axiom of choice: If the well-ordering theorem holds, then so does the axiom of choice.

Cardinal numbers: An ordinal number α is a cardinal number if α is not equinumerous with β for all ordinals $\beta < \alpha$.

Cardinality: Let X be a set. The cardinal number / cardinality of X is the unique cardinal which is equinumerous with X, denoted by $|X|$.

The Class of Cardinals: Finite ordinals are natural numbers. The class of infinite cardinals is constructed by transfinite recursion as:

- $\aleph_0 = \omega$
- $\aleph_{\alpha+1} = \aleph(\aleph_\alpha)$, namely the Hartogs number of $\aleph_\alpha$ for all ordinals α.
- $\aleph_\gamma = \sup\{\aleph_\theta : \theta < \gamma\}$ for all limit ordinals γ.

For all ordinals α, $\aleph_\alpha$ is a cardinal number. A cardinal number λ is a successor cardinal if $\lambda = \kappa^+$, and is a limit cardinal otherwise.

Cardinal Arithmetic: Given two cardinal numbers κ & λ, we have

- Addition: $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$
- Multiplication: $\kappa\lambda = |\kappa \times \lambda|$
- Exponentiation: $\kappa^\lambda = |{}^\lambda\kappa|$

"Rules" for cardinal arithmetic:

1) $\aleph_\alpha \cdot \aleph_\alpha = |\aleph_\alpha \times \aleph_\alpha| = \aleph_\alpha$ for all ordinals α.

2) For infinite cardinals κ & λ, $\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}$

κ many κ sets: Let κ be an infinite cardinal and $\{X_\alpha\}_{\alpha\in\kappa}$ be an indexed system such that $\forall\alpha$ $|X_\alpha|\leq\kappa$. Then

$$\left|\bigcup_{\alpha\in\kappa} X_\alpha\right|\leq\kappa$$

This is the generalization of "countable union of countable sets is countable."

Continuum Hypothesis: There is no set X such that $|\mathbb{N}|<|X|<|\mathbb{R}|$

The cardinality of the set of real numbers, denoted by $\mathfrak{c}$, is equal to the cardinal $2^{\aleph_0}$, and so the Continuum Hypothesis says that

$$\aleph_1 = 2^{\aleph_0} = \mathfrak{c}$$

Bet Numbers: By transfinite recursion, define the class of bet numbers:

- $\beth_0 = \aleph_0$
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$ for all ordinals α.
- $\beth_\gamma = \sup\{\beth_\beta : \beta<\gamma\}$ for all limit ordinals γ.

In this notation, the Continuum Hypothesis becomes

$$\aleph_1 = \beth_1$$

Generalized Continuum Hypothesis: For all ordinals α,

$$\aleph_\alpha = \beth_\alpha$$

It is also asserted that

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}$$

Independence from ZFC: CH & GCH are independent from the axioms of ZFC, given that they are consistent.

Cantor's theorem revisited: For any cardinal κ,

$$\kappa<2^\kappa \qquad (\,|X|<|\mathcal{P}(X)|\,)$$

Cofinal: Let $(P,\leq)$ be a partially ordered set. A subset $A\subseteq P$ is a cofinal of P if for all $p\in P$, there exists some $q\in A$ such that $p\leq q$.

Cofinality: Let α be an ordinal. Cofinality of α, namely $cf(\alpha)$ is the least ordinal λ such that there exists a function $f:\lambda\to\alpha$ such that the range of f is cofinal in $(\alpha,\leq)$

König's Theorem: Let $\{\lambda_i\}_{i\in I}$ and $\{\kappa_i\}_{i\in I}$ be indexed systems of cardinals such that $\lambda_i < \kappa_i$ for all $i \in I$. Then we have

$$\sum_{i\in I} \lambda_i < \prod_{i\in I} \kappa_i$$

As a consequence, we have that for any infinite cardinal κ,

$$\kappa < \kappa^{cf(\kappa)}$$

$$\kappa < cf(2^\kappa)$$

Regular and Singular Cardinals: A cardinal κ is said to be a regular cardinal if $cf(\kappa) = \kappa$. Otherwise, it is a singular cardinal.

Successor cardinals are regular.

Cardinal exponentiation under GCH: According to ZFC + GCH,

$$\kappa^\lambda = \begin{cases} \lambda^+ & \text{if } \kappa \leq \lambda \\ \kappa^+ & \text{if } cf(\kappa) \leq \lambda \leq \kappa \\ \kappa & \text{if } \lambda < cf(\kappa) \end{cases}$$